Algebraic groups

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These are notes for a class on algebraic groups taught by Rajesh Kulkarni at Michigan State University during summer 2019. The following books were the main sources for the class.

- *Conjugacy classes in algebraic groups*, Steinberg [6]
- *Linear algebraic groups*, T.A. Springer [5]
- *Linear algebraic groups*, Humphreys [3]
- *Linear algebraic groups*, Borel [2]

All errors in these notes are the responsibility of the note taker, Joshua Ruiter.

Contents

1 Affine algebraic varieties

1.1 Basic definitions

Throughout, let K be an algebraically closed field.

Definition 1.1. **Affine n-space** is K^n, also denoted $\mathbb{A}^n$.

Definition 1.2. An **algebraic set** in K^n (or $\mathbb{A}^n$) is the set of common zeros of a collection of polynomials in the polynomial ring $K[x_1, \ldots, x_n]$.

Definition 1.3. Let U be a set and let A be a K-algebra of functions $U \to K$, and let $x \in U$. The **evaluation map** associated to x is

$$\mathrm{ev}_x : A \to K \qquad \mathrm{ev}_x(f) = f(x)$$

This is a K-algebra homomorphism.

Definition 1.4. An **abstract affine algebraic variety** is a pair (U, A) where U is a set and A is a K-algebra of functions $U \to K$ satisfying the following three properties.

1. A is a finitely generated K-algebra.
2. A separates points of U. That is, given any two distinct points $x, y \in U$, there exists a function $f \in A$ such that $f(x) \neq f(y)$.
3. Every K-algebra homomorphism $\phi : A \to K$ is the evaluation map at some point $x \in U$. That is, $\phi = \mathrm{ev}_x$ for some $x \in U$, which is to say, $\phi(f) = f(x)$ for all $f \in A$.

Note that condition 2 implies that the point x in condition 3 is unique. Thus there is a bijection

$$\begin{aligned} \{\text{points in } U\} &\longleftrightarrow \{K\text{-algebra homomorphisms } A \to K\} \\ x &\longmapsto \mathrm{ev}_x \end{aligned}$$

We usually abbreviate and refer to an abstract affine algebraic variety just as an **affine variety**.

Example 1.5. The pair $(K^n, K[x_1, \ldots, x_n])$ is an affine variety.

Example 1.6. Let $V \subset K^n$ be an algebraic set. Let $K[x_1, \ldots, x_n]\big|_V$ denote the algebra of polynomial functions restricted to V. Then $(V, K[x_1, \ldots, x_n]\big|_V)$ is an affine variety. (This is not immediate, it requires the Hilbert Nullstellensatz.)

Example 1.7. Let A be a finitely generated K-algebra with no nilpotent elements. Because A is finitely generated, for some $n > 0$, there exists a surjection $K[x_1, \ldots, x_n] \twoheadrightarrow A$. Let $I \subset K[x_1, \ldots, x_n]$ be the kernel of this map, and let

$$V = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, \ldots, a_n) = 0 \; \forall f \in I\}$$

Then (V, A) is an affine variety. (As with the previous example, this is not immediate, but requires the Nullstellensatz.)

1.2 Morphisms

Definition 1.8. Let (U, A) and (V, B) be affine varieties. Temporarily denote the K-algebra of all functions $U \to K$ by $\widetilde{A}$. Given a set map $f : U \to V$, the **pullback** of f is the map

$$f^* : B \to \widetilde{A} \qquad \phi \mapsto \phi \circ f \qquad f^*(\phi) = \phi \circ f$$

A **morphism of affine varieties** $f : (U, A) \to (V, B)$ is a set map $f : U \to V$ such that the image of f^* is contained in A, that is, f^* is a map $B \to A$.

Remark 1.9. Let $f : (U, A) \to (V, B)$ be a morphism of affine varieties, and let $u \in U$. Then $f(u)$ is a point in V, so it corresponds to an evaluation map $\mathrm{ev}_{f(u)} : V \to K$. In particular, for $\phi \in V$, we have

$$\mathrm{ev}_{f(u)}(\phi) = \phi(f(u)) = (f^*\phi)(u) = \mathrm{ev}_u \circ f^*(\phi)$$

Thus

$$\mathrm{ev}_{f(u)} = \mathrm{ev}_u \circ f^*$$

One consequence of this is that f is determined by f^*.

Remark 1.10. Affine varieties along with their morphisms as we have defined them form a category. This is just a fancy way of saying that the composition of two morphisms is a morphism.

Lemma 1.11. *Let (U, A) be an affine variety. Every $f \in A$ is a morphism of varieties $f : (U, A) \to (K, K[x])$.*

Proof. We just need to check that f^* is a map $K[x] \to A$. Let $g \in K[x]$, viewed as a map $K \to K, \lambda \mapsto g(\lambda)$. So we have a map $f^*(g) : U \to K$, and we just need to verify that $f^*(g) \in A$.

$$f^*(g)(u) = g(f(u))$$

Since g is a polynomial, $g(f(u))$ is a polynomial in $f(u)$, so $f^*(g)$ is a polynomial in f. Since $f \in A$ and A is a K-algebra, $f^*(g) \in A$, hence f is a morphism of varieties. □

1.3 Subvarieties

Definition 1.12. Let (V, A) be an affine variety, and let $V' \subset V$. Let

$$A|_{V'} = \{f|_{V'} \mid f \in A\}$$

Note that we have a morphism of K-algebras

$$\mathrm{res} : A \to A|_{V'} \qquad f \mapsto f|_{V'}$$

If $(V', A|_{V'})$ is an affine variety, we call it a **subvariety** of (V, A).

Lemma 1.13. *Let (V, A) be an affine variety, and $V' \subset V$. Then $(V', A|_{V'})$ is a subvariety if and only if V' is a set of zeros (in V) of a collection of elements of A.*

Proof. ($\Longleftarrow$) Suppose V' is the set of common zeros of some subset $I \subset A$. If V' is any subset of V, then it is clear that $A|_{V'}$ is finitely generated. It is also easy to see that $A|_{V'}$ separates points of V', as follows: if $x, y \in V'$ are distinct points in V', there exists $f \in A$ so that $f(x) \neq f(y)$, and then $f|_{V'}(x) \neq f|_{V'}(y)$.

For the third property, let $\phi : A|_{V'} \to K$ be a K-algebra homomorphism. Then the composition $\phi \circ \text{res} : A \to K$ is a K-algebra homomorphism, so there exists $x \in V$ such that $\phi \circ \text{res} = \text{ev}_x$. That is, for $f \in A$,

$$\phi\left(f|_{V'}\right) = \phi \circ \text{res}(f) = \text{ev}_x(f) = f(x)$$

All that remains to show is that $x \in V'$. Let $g \in I$, that is, $\text{res}\, g = g|_{V'} = 0$. Then

$$g(x) = \phi\left(g|_{V'}\right) = \phi(0) = 0$$

Thus any function which vanishes on V' vanishes at x. Since by definition V' is the set of common zeros of A', $x \in V'$.

($\Longrightarrow$) Conversely, suppose that $(V', A|_{V'})$ is a subvariety. Define

$$I = \{f \in A \mid f|_{V'} = 0\}$$

It is clear that V' is contained in the common zero set of I; we just need to check that I has no "extra" common zeros outside V', that is, we need to show that if $x \in V$ such that $f(x) = 0$ for all $f \in I$, then $x \in V'$. So let $x \in V$ with $f(x) = 0$ for all $f \in I$. Define

$$\phi : A|_{V'} \to K \qquad f|_{V'} \mapsto f(x)$$

We need to verify that this is well-defined. If $f, g \in A$ such that $f|_{V'} = g|_{V'}$, then $(f-g)|_{V'} = 0$, so by definition of x,

$$\phi\left(f|_{V'}\right) - \phi\left(g|_{V'}\right) = f(x) - g(x) = (f-g)(x) = 0$$

Hence ϕ is well defined (it is also clearly a K-algebra homomorphism). Since $(V', A|_{V'})$ is a variety, ϕ is an evaluation map at some $y \in V'$. It also makes the following diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{\text{ev}_x} & K \\ {\scriptstyle \text{res}}\downarrow & \nearrow_{\phi = \text{ev}_y} & \\ A|_{V'} & & \end{array}$$

We claim $x = y$. If not, then by the separation axiom, there exists $f \in A$ such that $f(x) \neq f(y)$. On the other hand,

$$f(y) = f|_{V'}(y) = \text{ev}_y \circ \text{res}(f) = \text{ev}_x(f) = f(x)$$

which is a contradiction. Thus $x = y$, so $x \in V'$. $\square$

Definition 1.14. Let (V, A) be an affine variety. Let $f \in A$. The **principal open affine subset** associated to f is

$$V_f = \{x \in V \mid f(x) \neq 0\}$$

We haven't yet defined a topology on V, but this will be an open set once we do. Alternatively, one can take the sets V_f as basic open subsets and define the topology in this way. It is clear that the sets V_f cover V. All that is needed is to check that for every pair V_f, V_g, and every element $x \in V_f \cap V_g$, there is another principal open affine subset V_h such that $x \in V_h \subset V_f \cap V_g$.

Example 1.15. The pair (V_f, A_f) is an affine variety, where A_f denotes the localization of A at the subset $\{1, f, f^2, \ldots\}$. Note that

$$A_f \cong A[z]/(zf - 1)$$

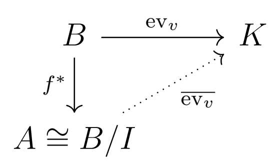
Lemma 1.16. *Let $f : (U, A) \to (V, B)$ be a morphism of affine varieties.*

1. *Let $I = \ker f^*$. Then*

$$f(U) = \{v \in V \mid \mathrm{ev}_v(g) = 0,\ \forall g \in I\}$$

2. *If $f^* : B \to A$ is surjective, then $(f(U), B|_{f(U)})$ is a subvariety of (V, B) and $f : (U, A) \to (f(U), B|_{f(U)})$ is an isomorphism. In particular, $f : U \to V$ is injective as a set map.*

Proof. (1) The inclusion $\subset$ follows from the fact that $\mathrm{ev}_{f(u)} = \mathrm{ev}_u \circ f^*$, noted in Remark 1.9. For the reverse inclusion, suppose $v \in V$ such that $\mathrm{ev}_v(g) = 0$ for all $g \in I$. Then ev_v factors through A, as in the following diagram.



By property 3 of affine varieties, $\overline{\mathrm{ev}_v} = \mathrm{ev}_u$ for some $u \in U$. But then

$$e_{f(u)} = e_u \circ f^* = \mathrm{ev}_v$$

so by uniqueness, $v = f(u)$.

(2) By (1) and Lemma 1.13, $(f(U), B|_{f(U)})$ is a subvariety of (V, B). Note that $B|_{f(U)} \cong B/I$. Since $f^* : A \to B/I$ is an isomorphism, let $g^* : B/I \to A$ be its inverse, which defines a morphism $g : (f(U), B/I) \to (U, A)$, and it is clear that g, f are inverses. $\square$

Proposition 1.17. *Any affine variety is isomorphic to a subvariety of $(\mathbb{A}^n, K[x_1, \ldots, x_n])$ for some n.*

Proof. This is immediate from Lemma 1.16. Give (U, A), take a surjective morphism $K[x_1, \ldots, x_n] \to A$, and us the lemma to obtain a morphism $(U, A) \to (\mathbb{A}^n, K[x_1, \ldots, x_n])$ which is an isomorphism onto its image. $\square$

1.4 Products

Definition 1.18. Let (U, A) and (V, B) be affine varieties. We define their product to be $(U \times V, A \otimes_K B)$. Henceforth we will usually omit the subscript K in tensor products when it will not create confusion.

Lemma 1.19. *The product $(U \times V, A \otimes B)$ is an affine variety.*

Proof. It is clear that $A \otimes B$ is finitely generated. For the separation axiom, let (x_1, y_1) and (x_2, y_2) be distinct points in $U \times V$. Without loss of generality, assume $x_1 \neq x_2$. Then there exists $f \in A$ such that $f(x_1) \neq f(x_2)$, so $f \otimes 1$ separates the points (x_1, y_1) and (x_2, y_2). A parallel argument works if $y_1 \neq y_2$.

It just remains to show that any K-algebra homomorphism $A \otimes B \to K$ is an evaluation map. Let $\phi : A \otimes B \to K$ be a K-algebra homomorphism. Consider the K-algebra homomorphisms

$$\begin{aligned} A &\to A \otimes B & \quad a &\mapsto a \otimes 1 \\ B &\to A \otimes B & \quad b &\mapsto 1 \otimes b \end{aligned}$$

and the respective compositions with ϕ, which give maps

$$\begin{aligned} A &\to K & \quad a &\mapsto \phi(a \otimes 1) \\ B &\to K & \quad b &\mapsto \phi(1 \otimes b) \end{aligned}$$

These maps must be evaluation maps, so there exist unique $x \in U, y \in V$ such that

$$\begin{aligned} \mathrm{ev}_x &: A \to K & \quad \mathrm{ev}_x(a) &= \phi(a \otimes 1) \\ \mathrm{ev}_y &: B \to K & \quad \mathrm{ev}_y(b) &= \phi(1 \otimes b) \end{aligned}$$

Consider the map

$$\psi : A \times B \to K \qquad (a, b) \mapsto \mathrm{ev}_{(x,y)}(a \otimes b) = \mathrm{ev}_x(a)\,\mathrm{ev}_y(b)$$

This is a K-balanced map, so by the universal property of the tensor product, there is a unique map $A \otimes B \to K$ making the following diagram commute. By uniqueness, this map is $\mathrm{ev}_{(x,y)}$.

$$\begin{array}{ccc} A \times B & \xrightarrow{\otimes} & A \otimes B \\ & \searrow^{\psi} & \downarrow{\scriptstyle \mathrm{ev}_{(x,y)}} \\ & & K \end{array}$$

We want to show that $\phi = \mathrm{ev}_{(x,y)}$. Because of uniqueness, it suffices that to show that ϕ makes this diagram commute as well. At least for tensors of the form $a \otimes 1, 1 \otimes b \in A \otimes B$, it is clear that $\phi \circ \otimes$ agrees with ψ.

$$\phi(a \otimes 1) = \mathrm{ev}_x(a) = \mathrm{ev}_x(a)\,\mathrm{ev}_y(1) = \psi(a, 1)\phi(1 \otimes b) = \mathrm{ev}_y(b) = \mathrm{ev}_x(1)\,\mathrm{ev}_y(b) = \psi(1, b)$$

Since such tensors generate $A \otimes B$ as a K-algebra, this implies $\phi \circ \otimes = \psi$. $\square$

Remark 1.20. The product variety has the usual universal property of categorial products. There are canonical maps $\pi_1 : (U \times V, A \otimes B) \to (U, A)$ and $\pi_2 : (U \times V, A \otimes B) \to (V, B)$, and if (W, C) is an affine variety with morphisms $(W, C) \to (U, A)$ and $(W, C) \to (U, A)$, then there is a unique morphism $(W, C) \to (U \times V, A \otimes B)$ making the following diagram commute.

$$\begin{array}{ccccc} & & (W, C) & & \\ & \swarrow & \vdots & \searrow & \\ (U, A) & \xleftarrow{\pi_1} & (U \times V, A \otimes B) & \xrightarrow{\pi_2} & (V, B) \end{array}$$

Once we have defined topologies, it will make sense to note that π_1, π_2 are open maps (that is, the associated set maps $U \times V \to U$ and $U \times V \to V$ are open maps).

1.5 Tangent spaces

The material in this section was not covered in the course, but added by the note taker in retrospect.

The definitions in this section are not standard for varieties. In fact, it is difficult to find any modern sources on algebraic geometry which do not take the perspective of schemes and define tangent spaces in that language. Since these notes do not use the language of schemes, I wanted a more "affine" definition of tangent spaces.

I have modeled this on the definition of tangent spaces for real manifolds, which are defined in terms of derivations. Because I don't have a source for this, I might be "wrong," in the sense that these definitions do not agree with the standard scheme-theoretic definition of tangent spaces.

Definition 1.21. Let (V, A) be an abstract affine variety. Fix $v \in V$. A **derivation of A at v** is a K-linear map $D : A \to K$ satisfying the Leibniz rule satisfying

$$D(ab) = D(a)b(v) + a(v)D(b)$$

for all $a, b \in A$. Note the above is an equality in K. Set

$$\text{Der}_v = \{D : A \to K | D \text{ is a derivation at } v\}$$

Given two derivations D, D' of A at v, we define their sum by

$$(D + D')(a) = D(a) + D'(a)$$

and for $\lambda \in K$, we define

$$(\lambda D)(a) = \lambda(D(a))$$

This makes Der_v into a K-vector space. We define the **tangent space** of V at v to be Der_v. Henceforth, we will refer to the tangent space by T_vV instead of Der_v.

Remark 1.22. This definition of tangent space unfortunately lacks any sense of the geometry implied by the use of the word "tangent." But we're living in the world of abstract algebraic varieties, with emphasis on abstract and algebraic, so this is just what we have to deal with.

Definition 1.23. Let $\alpha : (U, K[U]) \to (V, K[V])$ be a morphism of algebraic varieties, and let $u \in U$. The **differential** of α at u is the induced map

$$(d\alpha)_u : T_uU \to T_{\alpha(u)}V \qquad D \mapsto (f \mapsto D(f \circ \alpha))$$

where $f \in K[V]$. We can also write this as

$$(d\alpha)_u(D)(f) = D(f \circ \alpha)$$

To help keep things straight, we give the following list of the maps above with their domains and ranges.

Location	Domain and range
$\alpha \in \text{Hom}(U, V)$	$\alpha : U \to V$
$D \in T_uU$	$D : K[U] \to K$
$f \in K[V]$	$f : V \to K$
$f \circ \alpha \in K[U]$	$f \circ \alpha : U \to K$
$D(f \circ \alpha) \in K$	
$(d\alpha)_u(D) \in T_{\alpha(u)}$	$(d\alpha)_u(D) : K[V] \to K$

To fulfill all righteousness, one should really verify that $(d\alpha)_u(D)$ is a derivation at $f(u)$, that is, checking that it is K-linear and satsifies the Leibniz rule. These aren't that hard to check, but I'm not feeling it right now.

Example 1.24. We give a somewhat pathological example of how the differential of a morphism might fail to be "well-behaved," in the sense that it is not what we expect. Let K be a field of characteristic $p > 0$ (still algebraically closed), and consider the affine line $\mathbb{A}^1$ over K. Then we have a morphism of varieties

$$\text{Frob} : \mathbb{A}^1 \to \mathbb{A}^1 \qquad x \mapsto x^p$$

Actually, this is a morphism of algebraic groups, but that won't make sense until later. In any case, we claim that for $v \in \mathbb{A}^1$, $(d\,\text{Frob})_v = 0$. Let $D \in T_v\mathbb{A}^1$ and $f \in K[\mathbb{A}^1] \cong K[x]$. Note that $f \circ \text{Frob}(x) = f(x^p) = f(x)^p$ so $f \circ \text{Frob} = f^p$.

$$(d\,\text{Frob})_v(D)(f) = D(f \circ \text{Frob}) = D(f^p)$$

Using the Leibniz rule repeatedly with a sprinkling of induction,

$$\begin{aligned} D(f^p) &= D(f)f^{p-1}(v) + f(v)D(f^{p-1}) \\ &= D(f)f^{p-1}(v) + f^{p-1}(v)D(f) + f^2(v)D(f) \\ &= \cdots \\ &= pD(f)f^{p-1}(v) \\ &= 0 \end{aligned}$$

Thus $(d\,\text{Frob})_v(D)(f) = 0$ for all D and all f, so $(d\,\text{Frob})_v(D) = 0$ for all D, so $(d\,\text{Frob})_v = 0$.

Remark 1.25. Let $\alpha : U \to V$ be a morphism of varieties. Provisionally, we might define the "tangent bundle" of U by

$$TU = \bigsqcup_{u \in U} T_u U$$

Ideally, we would make TU into a variety which is a sort of fiber bundle over U, and possible there should be some sort of gluing conditions, but we'll ignore this. At least, it is clear there is a "projection"

$$TU \to U \qquad x \in T_u U, x \mapsto u$$

Presumably, with the right setup, it would make sense to talk about the "total differential" of α, which should be something which restricts to $(d\alpha)_u$ on the fiber $T_u U$.

$$d\alpha : TU \to TV \qquad (d\alpha)|_{T_u U} = (d\alpha)_u$$

We're not going to do anything with this, but it's a nice way to talk about all the differentials at the same time.

2 Affine and linear algebraic groups

2.1 Definitions and examples

Definition 2.1. An **affine algebraic group** is a pair (G, A) such that (G, A) is an affine algebraic variety, G is a group, and the group operations

$$G \times G \xrightarrow{m} G \qquad (x, y) \mapsto xy$$
$$G \xrightarrow{i} G \qquad x \mapsto x^{-1}$$

are morphisms of affine varieties. That is, $m^* : A \to A \otimes A$ and $i^* : A \to A$.

Example 2.2. Let V be an n-dimensional vector space over K. Fixing a basis of V gives an isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}(n, K)$. Then $(GL(V), K[x_{11}, \ldots, x_{nn}]_{\mathrm{det}})$ is an affine algebraic group, as we now explain.

Think of the variables x_{ij} for $1 \leq i, j \leq n$ as tracking the n^2 entries of a matrix $(x_{ij}) \in \mathrm{GL}(n, K)$. The subscript det means that the polynomial ring $K[x_{ij}]$ has been localized at the multiplicative set $\{1, d, d^2, \ldots\}$ where d is the expression for the determinant as a polynomial in the x_{ij}.

Let us justify why the multiplication map $\mathrm{GL}(V) \times \mathrm{GL}(V) \to \mathrm{GL}(V)$ is a morphism of varieties, at least in cases of small values of n. We need to verify that m^* is a map

$$m^* : K[z_{ij}]_{\mathrm{det}} \to K[x_{ij}]_{\mathrm{det}} \otimes K[y_{ij}]_{\mathrm{det}}$$

In the case $n = 1$, m is just the multiplication map $K^\times \times K^\times \to K^\times$, so m^* is the map

$$m^* : K\left[z, \frac{1}{z}\right] \to K\left[x, \frac{1}{x}\right] \otimes K\left[y, \frac{1}{y}\right] \qquad z \mapsto x \otimes y$$

In the case $n = 2$, $m : \mathrm{GL}(2, K) \times \mathrm{GL}(2, K) \to \mathrm{GL}(2, K)$ is the map

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{pmatrix}$$

so $m^* : K[z_{ij}]_{\det} \to K[x_{ij}]_{\det} \otimes K[y_{ij}]_{\det}$ is the map

$$\begin{aligned} z_{11} &\mapsto x_{11} \otimes y_{11} + x_{12} \otimes y_{21} \\ z_{12} &\mapsto x_{11} \otimes y_{12} + x_{12} \otimes y_{22} \\ &\text{etc.} \end{aligned}$$

One can check that $z_{11}z_{22} - z_{12}z_{21}$ gets mapped to an invertible element in the tensor product, but this is tedious. By sufficient waving of hands, this generalizes to any n, so multiplication is a morphism of varieties. Inversion is a morphism of varieties basically because of Cramer's rule.

Definition 2.3. The case of $G = \mathrm{GL}(1, K)$ is denoted $\mathbb{G}_m$ (which is isomorphic to $K^\times$). When viewing K as an additive group, it is also an algebraic group, denoted $\mathbb{G}_a$.

Example 2.4. As above, let V be an n-dimensional K-vector space. Then $(\mathrm{SL}(V), K[x_{ij}]_{\det}/(\det -1))$ is a subvariety of $\mathrm{GL}(V)$.

Example 2.5. Let $D_n \subset \mathrm{GL}(n, K)$ be the subgroup of diagonal matrix, that is, $D_n \cong (K^\times)^n$. Then

$$\left(D_n, {}^{K[x_{ij}]_{\det}}\!\big/_{x_{ij}, i \neq j}\right)$$

is a subvariety of $\mathrm{GL}(V)$.

Definition 2.6. Let V be a finite dimensional K-vector space. A **linear algebraic group** is an affine subvariety of $\mathrm{GL}(V)$ which is also a subgroup.

2.2 Every algebraic group is linear

Remark 2.7. Let (G, A) be an affine algebraic group, so we have morphisms of varieties

$$m : G \times G \to G \qquad i : G \to G$$

which correspond to K-algebra homomorphisms

$$m^* : A \to A \otimes A \qquad i^* : A \to A$$

Fix $x \in G$, and consider the right multiplication map

$$\rho_x : G \to G \qquad y \mapsto m(y, x) = yx$$

which is an automorphism of G as a variety (though it is not a group homomorphism). Thus ρ_x induces a K-algebra automorphism of A

$$\rho_x^* : A \to A \qquad f \mapsto \rho_x^*(f) = (y \mapsto f(yx)) \qquad \rho_x^*(f)(y) = f \circ \rho_x(y) = f(yx)$$

Note that

$$\rho_{xy} = \rho_y \circ \rho_x \qquad (\rho_{xy})^* = \rho_x^* \circ \rho_y^*$$

so we have a homomorphism of groups

$$\rho^* : G \to \operatorname{Aut}(A) \qquad x \mapsto \rho_x^*$$

Alternately, repeating the same story with the left multiplication map

$$\lambda_x : G \to G \qquad y \mapsto xy$$

yields a group homomorphism

$$\lambda^* : G \to \operatorname{Aut}(A) \qquad x \mapsto \lambda_x^*$$

Let A be a K-algebra. Given a collection of elements in A, we can talk about the subalgebra of A that they generate, or we can talk about the K-vector subspace they generate (span), and these are not in general the same. The subalgebra generated by a subset contains the span, but the span may be strictly smaller. In fact, it may be much much smaller. For example, in the polynomial ring $K[x]$, the subalgebra generated by $1, x$ is all of $K[x]$, which is an infinite dimensional K-vector space, but the span of $1, x$ is just a two dimensional space. To distinguish these types of generation, we introduce some notation.

Definition 2.8. Let A be a K-algebra and let $S \subset A$ be a subset of elements. The subalgebra generated by S is denote $\langle S \rangle$, and the vector subspace generated by S is denote $\langle S \rangle_K$. Note that $\langle S \rangle_K \subset \langle S \rangle$, but the inclusion is often strict.

Lemma 2.9. *Let V be a K-vector space, and let*

$$v = \sum_{i=1}^{n} u_i \otimes w_i \in V \otimes V$$

such that n is minimal (v cannot be written as a sum of a smaller number of simple tensors). Then $\{u_1, \ldots, u_n\}$ and $\{w_1, \ldots, w_n\}$ are both linearly independent sets.

Proof. If some u_j can be written as a linear combination of the other u_i, then that relation can reduce the number of terms needed to write v. Apply the same argument for the w_i. □

Lemma 2.10. *Let (G, A) be an algebraic group, and keep the notation ρ_x^*, etc. from the previous remark. Fix $f \in A$, and let*

$$B = \langle \rho_x^* f \mid x \in G \rangle_K$$

Then B is finite dimensional over K.

Proof. If $f = 0$, there is nothing to prove, since in this case $B = 0$, so we may assume $f \neq 0$. We have $m^* f \in A \otimes A$, so write it as a sum of simple tensors,

$$m^* f = \sum_{i=1}^{n} g_i \otimes h_i$$

with $g_i, h_i \in A$ and n minimal. By the previous lemma, $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ are linearly independent sets. We claim that this implies there exist elements $x_1, \ldots, x_n \in G$ such that

$$\det \big(h_i(x_j)\big) \neq 0$$

Note that $h_i(x_j) \in K$, so this is just a matrix with entries in K. We prove our claim by induction. The case $n = 1$ is obvious. If we have $x_1, \ldots, x_{n-1} \in G$ such that the $(n-1) \times (n-1)$ matrix $\big(h_i(x_j)\big)$ has nonzero determinant, then view it as an $(n-1) \times (n-1)$ minor of the $n \times n$ matrix $\big(h_i(x_j)\big)$.

Now consider the following determinant, thought of as having entries in A, where X is a variable. The first $n-1$ columns have entries in K, though we think of K as a subset of A, so this is fine. We think of X as a variable which we can plug in values in G, so the following determinant is a function $G \to K$.

$$\begin{vmatrix} h_1(x_1) & \cdots & h_1(x_{n-1}) & h_1(X) \\ h_2(x_1) & \cdots & h_2(x_{n-1}) & h_2(X) \\ \vdots & \ddots & \vdots & \vdots \\ h_{n-1}(x_1) & \cdots & h_{n-1}(x_{n-1}) & h_{n-1}(X) \\ h_n(x_1) & \cdots & h_n(x_{n-1}) & h_n(X) \end{vmatrix}$$

If this function vanishes for all $X \in G$, then the functions $h_1, \ldots, h_n$ would be linearly dependent, which they are not, so there exists some $x_n \in G$ making this not vanish. This proves our claim. We return to proving that B is finite dimensional. For $y \in G$, we have

$$\rho_y^* f(x) = f(xy) = \sum_{i=1}^n g_i(x) h_i(y) = \left(\sum_{i=1}^n h_i(y) g_i \right)(x)$$

thus

$$\rho_y^* f = \sum_{i=1}^n h_i(y) g_i \tag{2.1}$$

In particular, for each j,

$$\rho_{x_j}^* f = \sum_{i=1}^n h_i(x_j) g_i$$

Since the determinant $\det \big(h_i(x_j)\big)$ is nonzero, it gives a change of basis between $\{g_1, \ldots, g_n\}$ and $\big\{\rho_{x_1}^* f, \ldots, \rho_{x_n}^* f\big\}$. In particular, they are bases for the same space, and since all of the $\rho_{x_j}^* f$ lie in B (by construction of B), we get $g_1, \ldots, g_n \in B$. By equation 2.1, the g_i span all of B, so $\{g_1, \ldots, g_n\}$ is a K-basis of B, and $\dim_K B = n$. □

Next we introduce a way of dealing with matrix groups which is closer to being coordinate-free. Recall that if V is a K-vector space, the **dual space** of V, denoted V^*, is the vector space of K-linear maps $V \to K$. Elements of V^* are sometimes called **linear functionals**.

Definition 2.11. Let V be a finite dimensional K-vector space. For $v \in V, a^* \in V^*$, the **matrix coefficient** associated to v, a^* is the function

$$m_{v,a^*} : \mathrm{GL}(V) \to K \qquad m_{v,a^*}(T) = a^*(T(v))$$

Note the following two immediate identites.

$$m_{v+v',a^*} = m_{v,a^*} + m_{v',a^*} \qquad m_{v,a^*+b^*} = m_{v,a^*} + m_{v,b^*}$$

for $v, v' \in V, a^*, b^* \in V^*$. These identiies say that we have a bilinear map

$$V \times V^* \to \mathrm{Hom}_K(\mathrm{GL}(V), K) \qquad (v, a^*) \mapsto m_{v,a^*}$$

The Hom_K means homomorphisms of K-vector spaces. Note that this m has nothing to do with group multiplication $m : G \times G \to G$, it's just an unfortunate double usage of the letter m.

Remark 2.12. Let V be a finite dimensional K-vector space, and let $\widetilde{A}$ be the algebra of K-valued functions $\mathrm{GL}(V) \to K$. Let $A' \subset \widetilde{A}$ be the subalgebra generated by all m_{v,a^*}.

$$A' = \langle m_{v,a^*} \mid v \in V, a^* \in V^* \rangle \subset \widetilde{A}$$

Let $\det \in A'$ be the determinant function, and let $A = A'_{\det}$ (this subscript means localization at the subset $\{1, \det, \det^2, \ldots\}$). Then $(\mathrm{GL}(V), A)$ is an affine algebraic group. It is a "coordinate-free" version of the group $(\mathrm{GL}_n(K), K[x_{ij}]_{\det})$ introduced earlier.

We give some mild attempt to justify why $(\mathrm{GL}(V), A)$ is an affine algebraic group. The main point is the at a K-algebra homomorphism $A \to K$ amounts to choosing values for the generators m_{v,a^*} for all $v \in V, a^* \in V^*$. Given values for m_{v,a^*}, using the equation

$$m_{v,a^*}(T) = a^*(T(v))$$

for all v, a^*, this determines T. So a K-algebra homomorphism $A \to K$ determines an element $T \in \mathrm{GL}(V)$.

Lemma 2.13. *Let (G, A) be an affine algebraic group, and let $B \subset A$ be a finite dimensional subspace that is invariant under ρ_G^* [1]. Define*

$$\alpha : G \to \mathrm{GL}(B) \qquad x \mapsto \alpha(x) = \rho_x^*|_B$$

1. *Then α is a morphism of affine algebraic groups [2].*

2. *The matrix coefficients m_{v,a^*} of G pulled back via α span the same linear space as $\lambda_G^*(B)$. That is,*

$$\langle m_{v,a^*} \circ \alpha \mid v \in V, a^* \in V^* \rangle_K = \langle \lambda_G^*(B) \rangle_K = \langle \lambda_x^*(v) \mid x \in G, v \in B \rangle_K$$

Proof. (1) Let $v \in B, a^* \in B^*$, and consider the matrix coefficient

$$m_{v,a^*} : \mathrm{GL}(B) \to K \qquad T \mapsto a^*(T(v))$$

[1] This means that for $x \in G$, $\rho_x^* : A \to A$ maps B to itself.

[2] $\mathrm{GL}(B)$ has the associated algebra from the previous remark, generated by matrix coefficients m_{v,a^*} for $v \in B, a^* \in B^*$

We also have multiplication $m : G \times G \to G$ and $m^* : A \to A \otimes A$. For $v \in B$, write $m^*(v)$ as

$$m^*(v) = \sum_{i=1}^{n} g_i \otimes h_i$$

with $g_i, h_i \in A$ and with n minimal. Then $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ are linearly independent, and from previous work (Lemma 2.10 and the proof of that lemma) we know that $\{g_1, \dots, g_n\}$ is a basis of $\langle \rho_y^*(v) \mid y \in G, v \in B\rangle_K$ and $\{h_1, \dots, h_n\}$ is a basis of $\langle \lambda_y^*(v) \mid y \in G, v \in B\rangle_K$.

To show that α is a morphism of algebraic groups, it suffices to show that α^* gives a K-algebra homomorphism from the algebra associated to $\mathrm{GL}(B)$ to A, that is, it suffices to show that each matrix coefficient m_{v,a^*} (for $v \in B, a^* \in B^*$) pulls back to a regular function on G (an element of A), and that α^* maps the determinant to an invertible element, which is to say, we must verify that $\frac{1}{\det}$ pulls back to a regular function on G as well. Now let $y \in G, v \in B, a^* \in B^*$. As in the Lemma 2.10,

$$\alpha^*(m_{v,a^*})(y) = m_{v,a^*} \circ \alpha(y) = m_{v,a^*}(\alpha(y)) = a^*\big(\alpha(y)(v)\big) = a^*\big(\rho_y^*|_B(v)\big) = a^*\left(\sum_{i=1}^{n} h_i(y) g_i\right)$$

so

$$m_{v,a^*} \circ \alpha = \sum_{i=1}^{n} a^*(g_i) h_i \tag{2.2}$$

Thus $m_{v,a^*} \circ \alpha$ is a linear combination of the h_i, which lie in A. So m_{v,a^*} pulls back to a regular function. Now we check that $\frac{1}{\det}$ also pulls back to a regular function. Let $y \in G$. Then

$$\frac{1}{\det} \circ \alpha(y) = \det(\alpha(y^{-1})) = \det \circ \alpha \circ i(y)$$

where $i : G \to G$ is the inversion map. Thus $\frac{1}{\det} \circ \alpha = \det \circ \alpha \circ i$, which is a regular function on G, so this completes the argument that α is a morphism of varieties. Now for (2), we need to show that

$$\langle m_{v,a^*} \circ \alpha\rangle_K = \langle \lambda_G^*(B)\rangle_K$$

By equation 2.2, the left side is equal to the span of $\{h_1, \dots, h_n\}$. So to show the inclusion $\subset$ it suffices to show $h_i \in \langle \lambda_G^*(B)\rangle_K$. By definition of g_i, h_i,

$$m^*(v) = \sum_{i=1}^{n} g_i \otimes h_i \implies v(yx) = \sum_{i=1}^{n} g_i(y) h_i(x) = \left(\sum g_i(y) h_i\right)(x)$$

so

$$\lambda_y^*(v) = \sum_{i=1}^{n} g_i(y) h_i \implies h_i \in \langle \lambda_G^*(B)\rangle_K \tag{2.3}$$

Thus

$$\langle m_{v,a^*} \circ \alpha\rangle_K \subset \langle \lambda_G^*(B)\rangle_K$$

Also by equation 2.3, $\lambda_y^*(v)$ is in the span of the h_i, so the reverse inclusion also holds. □

Proposition 2.14. *Every affine algebraic group is (isomorphic to) a linear algebraic group.*

Proof. Let (G, A) be an affine algebraic group. Let $\{1 = f_1, f_2, \ldots, f_n\}$ be a fintie set of generators for A as a K-algebra. Let

$$B = \langle \rho_x^* f_i \mid i = 1, \ldots, n, x \in G \rangle_K$$

By Lemma 2.10, for a fixed i, the span of $\rho_x^* f_i$ for $x \in G$ is finite dimensional, and B is a union of finitely many of those, so B is finite dimensional. Also, it is clear that B is invariant under ρ_G^*. Then by Lemma 2.13, we have a homomorphism of affine algebraic groups

$$\alpha : G \to \mathrm{GL}(B)$$

By the second part of Lemma 2.13,

$$\langle \alpha^*(m_{v,a^*}) = m_{v,a^*} \circ \alpha \rangle_K = \langle \lambda_G^*(B) \rangle_K$$

Let W denotes this subspace of A. So the image of α^* is a K-subalgebra of A containing W. But also, each $f_i \in W$, since

$$\lambda_e^*(f_i) = f_i \circ \lambda_e = f_i \in \lambda_G^*(B) \subset W$$

where $e \in G$ is the identity. Thus $\operatorname{im} \alpha^*$ is a subalgebra of A containing a full set of generators $f_1, \ldots, f_n$ of A, so $\operatorname{im} \alpha^* = A$. Then by Lemma 1.16, α gives an isomorphism of G with its image, so G is (isomorphic to) a subvariety of $\mathrm{GL}(B)$. □

2.3 Zariski topology

Definition 2.15. Let (V, A) be an affine variety. Let

$$\mathcal{F} = \{Z \subset V \mid Z \text{ is a set of common zeros of a collection of elements in } A\}$$

We claim that $\mathcal{F}$ satisfies the axioms to be the collection of closed sets for a topology on V. For the sake of notation, for $f \in A$, let $Z(f)$ denote the set of zeros of f. It is clear that $\emptyset, V \in \mathcal{F}$. Given a subset $\{Z_\alpha\} \subset \mathcal{F}$, the intersection $\bigcap_\alpha Z_\alpha$ is the set of zeros of all elements of all the elements defining each Z_α, so $\bigcap_\alpha Z_\alpha \in \mathcal{F}$. Finally, given $F_1, F_2 \in \mathcal{F}$, write each as

$$F_1 = \bigcap_\alpha Z(f_i^\alpha) \qquad F_2 = \bigcap_\alpha Z(f_2^\alpha)$$

then

$$F_1 \cup F_2 = \bigcap_{\alpha,\beta} Z(f_1^\alpha f_2^\beta)$$

This last equality requires some proof, but I'm too lazy right now. So $\mathcal{F}$ satisfies the axioms, and the topology defined by taking $\mathcal{F}$ to be the closed sets on V is the **Zariski topology** on V.

Lemma 2.16. *Morphisms of affine varieties are continuous with respect to the Zariski topology.*

Proof. Let $\alpha : (U, A) \to (V, B)$ be a morphism of affine varieties. To show that it is continuous, it suffices to show that the preimage of a closed set is closed. Let $Z \subset V$ be a closed set, so Z is the common zero locus of a collection $\{f_i\} \subset B$.

$$Z = \bigcap_i Z(f_i)$$

For each $f_i \in B$, then

$$\alpha^{-1}(Z(f_i)) = \{u \in U : f_i(\alpha(u)) = \alpha^* f_i(u) = 0\} = Z(\alpha^* f_i)$$

Thus

$$\alpha^{-1}(Z) = \alpha^{-1}\left(\bigcap_i Z(f_i)\right) = \bigcap_i \alpha^{-1}\left(Z(f_i)\right) = \bigcap_i Z(\alpha^* f_i)$$

Since α is a morphism of varieties, $\alpha^* f_i \in A$, so the preimage of the closed set Z is a closed set. $\square$

Lemma 2.17. *Let $\alpha : (U, A) \to (V, B)$ be a morphism of varieties such that the image of α is dense, that is, $\overline{\alpha(U)} = V$. Then $\alpha^* : B \to A$ is injective.*

Proof. Let $f \in B$ such that $\alpha^*(f) = 0$.

$$U \xrightarrow{\ \alpha\ } V \xrightarrow{\ f\ } K, \qquad U \xrightarrow{\ \alpha^* f\ } K$$

This means that for $u \in U$,

$$\alpha^*(f)(u) = f \circ \alpha(u) = 0$$

Thus $f|_{\alpha(U)} = 0$. By Lemma 1.11, f is a morphism of varieties, so it is continuous by the previous lemma. Since $\alpha(U)$ is dense in V, f is a continuous function which vanishes on a dense subset, which forces $f = 0$. Hence α^* is injective. $\square$

2.4 Irreducible components

Recall that the Hilbert basis theorem says that a finitely generated K-algebra is Noetherian (as a ring).

Definition 2.18. Let X be a topological space.

1. X is **Noetherian** if it satisfies the descending chain condition on closed subsets (or equivalently, the ascending chain condition on open subsets).

2. X is **irreducible** if it is not a union of two proper closed subsets. If X is not irreducible, it is **reducible**.

3. A subset $Y \subset X$ is **irreducible** if it is irreducible with respect to the subspace topology.

Remark 2.19. Irreducibility is not very interesting in many topological spaces. On a smooth manifold (over $\mathbb{R}$ or $\mathbb{C}$) with the usual Euclidean charts topology, the only irreducible subspaces are individual points. Loosely speaking, this is because those topologies have an abundance of closed sets. Irreducibility is only interesting when the topology is rather coarse, when it has "not so many" closed subsets. The Zariski topology on a variety is an example where there are not as many closed sets, so irreducibility becomes a useful notion.

Remark 2.20. In general an irreducible space is always connected, but the converse is not true (in the general case, it may happen).

Lemma 2.21. *Let (V, A) be an affine variety. Then V is irreducible if and only if A is an integral domain.*

Proof. ($\Longrightarrow$) Suppose A is not an integral domain, so there are zero divisors $f, g \in A$, with $f, g \neq 0$. That is, $V \to K, x \mapsto f(x)g(x)$ is the zero function. Then $V = Z(f) \cup Z(g)$ where $Z(f), Z(g)$ are proper (since $f, g \neq 0$) closed subsets, hence V is reducible.

($\Longleftarrow$) Conversely, if V is reducible, write it as $V = X \cup Y$ with X, Y proper closed subsets. Let $X = \bigcap_i Z(f_i), Y = \bigcap_j Z(g_j)$. Since A is Noetherian, the ideals $\langle f_i \rangle, \langle g_j \rangle \subset A$ are finitely generated, say by $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ respectively. Then $f_1 \cdots f_n|_X = 0$ and $g_1 \cdots g_m|_Y = 0$ are both not the zero function on all of V, but

$$f_1 \cdots f_n g_1 \cdots g_m = 0$$

on all of V. Which is to say, A has a zero divisor, so it is not an integral domain. □

Theorem 2.22. *Every Noetherian topological space can be expressed as a finite union of closed irreducible subsets. If the number of such sets used is minimal, then the irreducible subsets are maximal (among closed irreducible subsets of X).*

Proof. Skipped. Exercise from Atiyah-MacDonald, done in commutative algebra class last fall, see homework. □

Definition 2.23. Let X be a Noetherian topological space, and write it as a finite union of closed irreducible subsets, such that the number of them is minimal, so that they are maximal irreducible subsets. The resulting sets are called **irreducible components** of X.

Corollary 2.24. *Let (V, A) be an affine algebraic variety. Then V (with the Zariski topology) is Noetherian (by Hilbert basis theorem), so V has a decomposition into irreducible components.*

Proof. This follows from the correspondence between radical ideals of A and closed subsets of V, and the fact that prime ideals correspond to irreducible subets. □

Proposition 2.25 (Components of an algebraic group)**.** *Let (G, A) be an affine algebraic group and let $G^0 \subset G$ be the irreducible component containing the identity.*

1. *The irreducible components of G are pairwise disjoint, and are the connected components of G.*

2. *G^0 is a normal subgroup of finite index, which is also a subvariety.*

3. *The irreducible components of G are the cosets of G^0, so they are all isomorphic as affine varieties.*

Proof. (1) Let G_1, G_2 be irreducible components of G, and suppose $x \in G_1 \cap G_2$. Any automorphism of G (as a variety) permutes irreducible components, so by translating x to any $x' \in G$ (which we can do because G acts transitively on itself) we see that every element of G is in at least two irreducible components. We can also write G as a minimal union of irreducible components $G = X_1 \cup X_2 \cup \cdots \cup X_n$, but then every element of X_n is contained in some other component by what we just proved, contradicting minimality. So no element can be in two components. Since each irreducible component is also connected, pairwise disjointness implies that they are the connected components of G.

(2) Let $x \in G^0$. Then xG^0 is also an irreducible component, and $x \in xG^0$, so $xG^0 = G^0$, so G^0 is closed unde rmultiplication. Similarly, $(G^0)^{-1}$ is an irreducible component containing the identity, so it is G^0. Thus G^0 is a subgroup, and the irreducible components of G are the cosets, proving (3). G^0 has finite index because G has only finitely many irreducible components. Also G^0 is an affine subvariety, since it is closed (it is an irreducible component). It is normal because for $y \in G$, yG^0y^{-1} is an irreducible component containing the identity, so it must be G^0. □

Example 2.26. Let K be an algebraically closed field, as always. $\mathrm{GL}(n, K)$ and $\mathrm{SL}(n, K)$ are irreducible algebraic groups. (Maybe proved later?)

As a somewhat irrelevant contrast, consider the Lie group $\mathrm{GL}(n, \mathbb{R})$. It has two connected components, the one containing the identity which is matrices with positive determinant, and the other component is matrices with negative determinant. Don't get too hung up on this comparison, though, because as a Lie group this has a different topology than $\mathrm{GL}(n, K)$ with the Zariski topology.

Definition 2.27. Let V be an n-dimensional K-vector space with a nondegenerate symmetric bilinear form $B : V \times V \to V$. Define the **orthogonal group**

$$\mathrm{O}(V, B) = \{T \in \mathrm{GL}(V) : B(T(v), T(w)) = B(v, w),\ \forall v, w \in V\}$$

Alternately, we can think of this in terms of matrices. Fix a basis of V, so that the bilinear form B can be represented by a matrix $\widetilde{B} \in \mathrm{Mat}(n, K)$, and the bilinear form is given by

$$B : V \times V \to V \qquad (v, w) \mapsto w^t \widetilde{B} v$$

where the superscript t denotes tranpsoe. We can also represent $T \in \mathrm{GL}(V)$ as a matrix $\widetilde{T} \in \mathrm{GL}(n, K)$, that is,

$$T : V \to V \qquad v \mapsto \widetilde{T} v$$

In this language, the condition $B(T(v), T(w)) = B(v, w)$ becomes

$$(\widetilde{T} w)^t \widetilde{B} (\widetilde{T} v) = w^t \widetilde{T}^t \widetilde{B} \widetilde{T} v = w^t \widetilde{B} v$$

for all $v, w \in V$, which is equivalent to the condition

$$\widetilde{T}^t \widetilde{B} \widetilde{T} = \widetilde{B}$$

So we get a version of $\mathrm{O}(V, B)$ in terms of coordinates, which we denote $\mathrm{O}(n, K, B)$.

$$\mathrm{O}(V, B) \cong \mathrm{O}(n, K, B) = \left\{T \in \mathrm{GL}(n, K) : T^t B T = B\right\}$$

Frequently B and K are understood, so this is just written $\mathrm{O}(n)$ or O_n.

Definition 2.28. The **special orthogonal group** is the subgroup of O_n with determimant one.

$$\mathrm{SO}_n = \{x \in \mathrm{O}_n : \det x = 1\}$$

Example 2.29. O_n is reducible into two components, with SO_n being the irreducible component containing the identity. (Maybe proved later?)

Remark 2.30. Let (G, A) be an affine algebraic group.

1. G^0 is the smallest subgroup of G with finite index. (This is easy later, using dimension.)
2. Any closed subgroup of finite index in G is open.
3. If $S \subset G$ is closed under multiplication, then S is a subgroup.

Proofs of the above are left as exercises.

3 Group actions on varieties

In what follows, we often refer to an affine variety (V, A) just as V, with the associated algebra A understood. When we need to refer to the algebra associated to a variety V, we denote it by $K[V]$.

Definition 3.1. Let G be an affine algebraic group, and let V be an affine variety. G acts on V **as an algebraic group** if there is a group action of G on V such that the structure morphism

$$G \times V \to V \qquad (g, v) \mapsto gv$$

is a morphism of varieties. In the future, when we say, "Let G be an algebraic group acting on a variety V," we mean that it is acting as an algebraic group, not just as a group.

In order to obtain some basic results about algebraic groups acting on varieties, we pursue a technical lemma involving some commutative algebra.

3.1 Thick subsets of varieties

Definition 3.2. Let V be an affine variety, and let $U \subset V$ be a subset. U is a **thick** subset of V if U is irreducible, and U contains a dense open subset of $\overline{U}$.

Note that U being thick does not have much to do with the ambient space V. Before we prove the main lemma, we cite a result from commutative algebra.

Lemma 3.3. *Let $B \subset A$ be integral domains such that A is finitely generated as a B-algebra, and let K be an algebraically closed field. Let $f \in A, f \neq 0$. Then there exists $g \in B, g \neq 0$ such that for any homomorphism*

$$\alpha : B \to K$$

there is an extension $\widetilde{\alpha} : A \to K$, that is, a morphism making the following triangle commute, such that $\widetilde{\alpha}(f) \neq 0$.

$$\begin{array}{ccc} A & & \\ \uparrow & \overset{\widetilde{\alpha}}{\searrow} & \\ B & \xrightarrow[\alpha]{} & K \end{array}$$

Proof. Probably somewhere in Atiyah-MacDonald [1]. □

Now for the main lemma. From first reading the statement of the lemma, it seems like it should be a straightforward argument, but it is not so easy.

Lemma 3.4 (Thickness lemma)**.** *Let U, V be affine varieties and let $\alpha : U \to V$ be a morphism of varieties. Let $U' \subset U$ be a thick subset. Then $\alpha(U')$ is a thick subset of V.*

Proof. First we make three reductions. First, if the result holds in the case where $U = \overline{U'}$, then $\alpha(U')$ is a thick subset of a subset of V, so it is stil a thick subset of V, so the general case follows. So without loss of generality, we assume $U = \overline{U'}$, so U is irreducible because U' is thick.

Since U' contains a dense open subset of U, it contains a principal open subset $U_f = \{x \in U : f(x) \neq 0\}$ (since they give a basis for the topology). Now we make our second reduction: we may assume $U' = U_f$, since if the image of under α is thick for a principal open subset, then the image of the larger set U' is also thick.

As a third reduction, we may assume $V = \overline{\alpha(U')}$, since if $\alpha(U')$ is thick in $\overline{\alpha(U')}$, it is thick in any variety which as $\overline{\alpha(U')}$ as a subvariety. Since U' is irreducible, its image $\alpha(U')$ is irreducible, so $V = \overline{\alpha(U')}$ is irreducible (the closure of an irreducible set is irreducible).

Summarizing, we have reduced to proving the lemma in the case where we have a morphism $\alpha : U \to V$ with $U' = U_f \subset U$ for some $f \in K[U]$, where $U = \overline{U'}$ and $V = \overline{\alpha(U)}$, and U, V are irreducible. To show that $\alpha(U')$ is thick, it suffices to show that $\alpha(U')$ contains an open subset, since we already know it is irreducible.

Because the image of α is dense in V, by Lemma 2.17, $\alpha^* : K[V] \to K[U]$ is injective. Also recall that $K[V], K[U]$ are integral domains because U, V are irreducible. Now we apply Lemma 3.3 in the case $A = K[U], B = \alpha^* K[V]$. Since $U' = U_f$ is dense in U, $f \neq 0$. So by the lemma, there exists $g \in K[V]$ such that $\alpha^*(g) \neq 0$ and $g \neq 0$, satsifying the extension property of the lemma. Consider the following principal open subset of V.

$$V_g = \{v \in V : g(y) \neq 0\}$$

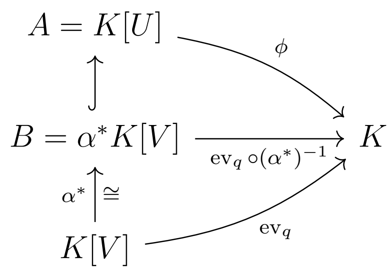
If we show that $V_g \subset \alpha(U')$, then the we are done. So let $q \in V_g \subset V$, so $g(q) \neq 0$. Consider the composition

$$\alpha^* K[V] \xrightarrow[\cong]{(\alpha^*)^{-1}} K[V] \xrightarrow{\mathrm{ev}_q} K$$

We know that

$$\mathrm{ev}_q \circ (\alpha^*)^{-1} \circ \alpha^*(g) = \mathrm{ev}_q(g) = g(q) \neq 0$$

So by the properties of g given by the conclusion of Lemma 3.3, there exists an extesion $\phi : A \to K$ making the following diagram commutes.



Since $A = K[U]$, the map ϕ is an evaluation map at some $p \in U$, so $\phi(f) = f(p) \neq 0$. Note that this implies $p \in U' = U_f$. We want to show that $\alpha(p) = q$. To do this, it suffices to show that $\mathrm{ev}_{\alpha(p)} = \mathrm{ev}_q$ as homomorphisms $K[V] \to K$. For $h \in K[V]$,

$$\mathrm{ev}_{\alpha(p)}(h) = h(\alpha(p)) = \alpha^*(h)(p) = \mathrm{ev}_p \circ \alpha^*(h) = \phi \circ \alpha^*(h) = \mathrm{ev}_q \circ (\alpha^*)^{-1} \circ \alpha^*(h) = \mathrm{ev}_q(h)$$

Thus $\mathrm{ev}_{\alpha(p)} = \mathrm{ev}_q$, so $q = \alpha(p)$. Thus $V_g \subset \alpha(U')$, and the proof is complete. □

3.2 Applications of the thickness lemma

Corollary 3.5. *Let G be a connected algebraic group acting on a variety V. Then every orbit is open in its closure.*

Proof. Let $G.v$ be the orbit of $v \in V$. Consider the morphism

$$\alpha : G \to V \qquad g \mapsto g.v$$

Since G is connected, it is irreducible, so it is thick in itself. So by the previous lemma, $\alpha(G) = G.v$ is thick in V. So $\alpha(G)$ contains an open subset of $\overline{\alpha(G)}$, call it U. So for some $g_0 \in G$, $g_0.v \in U$, so $v \in g_0^{-1}.U$. Then for all $g \in G$,

$$gg_0.v \in gg_0^{-1}.U \subset G.U \subset G.v$$

Thus

$$G.v = \bigcup_{g \in G} g.U$$

Since U is open in $\overline{G.v}$, the tranlates $g.U$ are open in $\overline{G.v}$, so $G.v$ is a union of open subsets, so it is open in $\overline{G.v}$. □

Corollary 3.6. *Let G be a connected algebraic group acting on a variety V, and let $v \in V$. Then*

$$\overline{G.v} = (G.v) \bigsqcup (\text{orbits of smaller dimension})$$

Proof. By the previous corollary, $G.v$ is open in $\overline{G.v}$, $\overline{G.v} \setminus G.v$ is a closed subset of the irreducible set $\overline{G.v}$, so it must have smaller dimension. $\square$

Corollary 3.7. *Let G be a connected algebraic group acting on a variety V. Then orbits of minimal dimension are closed, so closed orbits exist.*

Proof. If an orbit $G.v$ has minimal dimension, there are no orbits of smaller dimension, so by the previous corollary $\overline{G.v} \subset G.v$, which forces equality. $\square$

Remark 3.8. The previous corollaries do not actually require connectedness. This just takes a reduction step added to the first of the three corollaries.

Lemma 3.9. *Let $f : U \to V$ be a morphism of algebraic varieties, with U irreducible and $f(U)$ dense in V. Then for every $v \in V$,*

$$\dim f^{-1}(v) \geq \dim U - \dim V$$

and equality holds for a dense open subset of V.

Proof. Omitted. $\square$

Proposition 3.10. *Let $\alpha : G \to G'$ be a morphism of algebraic groups. Then*

1. *$\alpha(G)$ is closed in G'*
2. *$\alpha(G^0) = \alpha(G)^0$*
3. *$\dim G = \dim \ker \alpha + \dim \operatorname{im} \alpha$*

Proof. (1) G acts on G' via

$$G \times G' \to G' \qquad (a, b) \mapsto \alpha(a)b$$

which is an action as an algebraic group, because the map above factors as the following composition, which is a morphism of varieties.

$$\begin{array}{ccc} G \times G' & \longrightarrow & G' \\ {\scriptstyle \alpha\times\mathrm{Id}}\downarrow & \nearrow_{m} & \\ G' \times G' & & \end{array}$$

The orbits of this action are the cosets of $\alpha(G)$ in G'. So they are all isomorphic (as varieties). So they all have the same dimension, so they have minimal dimension. Then since orbits of minimal dimension are closed (Corollary 3.7), $\alpha(G)$ (and its cosets) are closed in G'.

(Alternate proof of (1)) Consider the action of G^0 on G' via α as in the first proof. So $\alpha(G)^0$ is open in its closure, since it is an orbit. Also $\alpha(G^0)$ and hence $\overline{\alpha(G^0)}$ are irreducible. For any $x \in \overline{\alpha(G^0)}$,

$$x\alpha(G^0)^{-1} \cap \alpha(G^0) \neq \emptyset$$

since both are open (dense) subsets of $\overline{\alpha(G^0)}$. So for some $y, z \in \alpha(G^0)$,

$$xy^{-1} = z \qquad x = zy$$

so $x \in \alpha(G^0)$ so $\alpha(G^0) = \overline{\alpha(G^)}$. Then since $\alpha 9G)$ is the union of cosets $\alpha(G^0)$, $\alpha(G)$ is closed.

(2) Since $\alpha(G)$ is closed, it is an algebraic (sub)group of G'. It is clear that $\alpha(G^0) \subset \alpha(G)^0$ since the image of the irreducible subset G^0 must be irreducible, so it is contained in the irreducible component containing the identity of G'.

For the reverse inclusion, note that G^0 is a subgroup of G of finite index, so $\alpha(G^0)$ is a closed subgroup of $\alpha(G)$ of finite index. By Remark 2.30, a closed subgroup of finite index contains the identity component, so $\alpha(G^0) \supset \alpha(G)^0$. Thus they are equal.

(3) By applying Lemma 3.9 to the morphism $\alpha : G^0 \to \alpha(G^0) = \alpha(G)^0$, for every element $x' \in G'$,

$$\dim G^0 \leq \dim \alpha^{-1}(x') + \dim \alpha(G)^0$$

with equality holding on an open subset of G^0. Since the fibers $\alpha^{-1}(x')$ are all cosets of $\ker \alpha \cap G^0$, they all have the same dimension, so the equality holds for all x'. In particular, it holds for x' equal to the identity of G', so

$$\dim G^0 = \dim \left(\ker \alpha \cap G^0\right) + \dim \alpha(G)^0$$

Since G is a disjoint union of copies of G^0, $\dim G = \dim G^0$ and similarly $\dim \alpha(G)^0 = \dim \alpha(G) = \dim \operatorname{im} \alpha$, and $\dim(\ker \alpha \cap G^0) = \dim \ker \alpha$, so this is the desired equality in disguise.

$$\dim G = \dim \ker \alpha + \dim \operatorname{im} \alpha$$

□

Example 3.11. We give an example where (1) of Proposition 3.10 fails in the case of an algebraic group over a field which is NOT algebraically closed. Consider $\mathbb{R}^\times$ as an algebraic group over the field $\mathbb{R}$, and the morphism

$$\mathbb{R}^\times \to \mathbb{R}^\times \qquad x \mapsto x^2$$

This is a morphism of algebraic groups, but the image (which is $\mathbb{R}_{>0}$) is not closed (in the Zariski topology).

4 Linear algebra - Jordan decomposition for $\mathrm{GL}(V)$

Since we showed that every algebraic group is a subgroup of $\mathrm{GL}(V)$, it makes sense to study some linear algebra at this point.

4.1 Preliminaries

Fix an algebraically closed field K.

Definition 4.1. Let V be a finite dimensional K-vector space. A **scalar** element of $\mathrm{End}(V)$ is a morphism of the form $\lambda \, \mathrm{Id}$, with $\lambda \in K$. An element of $\mathrm{End}(V)$ which is not of this form is called **nonscalar**.

Definition 4.2. Let V be a finite dimensional K-vector space, and let $A \in \mathrm{End}(V)$. A is **semisimple** if the following equivalent conditions are satisfied.

1. A is diagonalizable.

2. There is a basis of V consisting of eigenvectors of A.

3. The minimal polynomial of A has distinct roots.

4. $K[A] \subset \mathrm{End}(V)$ is a semisimple K-algebra (every representation of $K[A]$ decomposes as a direct sum of irreducible representations).

A is **nilpotent** if there exists $n \geq 1$ such that $A^n = 0$, or equivalently if all eigenvalues of A are zero. A is **unipotent** if $A - I$ is nilpotent (or equivalently, all eigenvalues are one).

Remark 4.3. Let V be a finite dimensional K-vector space, and let $A \in \mathrm{End}(V)$. Suppose $U \subset V$ is an invariant subspace of A, and consider $A|_U \in \mathrm{End}(U)$. If any of the properties above (scalar, nonscalar, semisimple, nilpotent, unipotent) applies to A, then it also applies to $A|_U$.

Remark 4.4. If a linear map is both semisimple and nilpotent, then it is zero, since the minimal polynomial of a nilpotent linear map is x^n, and being semisimple forces no repeated roots.

Lemma 4.5. *Let V be a finite dimensional K-vector space, and let $S \subset \mathrm{End}(V)$ be a set of pairwise comuting endomorphisms. Then there is a basis of V such that the matrices representing all elements of S are in upper triangular form, and so that all the semisimple elements of S are diagonal.*

A shorter but less precise version of the previous statement is: Any set of pairwise commuting endomorphisms can be simultaneously put in upper triangular form.

Proof. (Note this proof is incomplete.)

Let $S \subset \mathrm{End}(V)$ be a set of pairwise commuting endomorphisms. We proceed by induction on $\dim V$. If $\dim V = 1$, then $\mathrm{End}(V) \cong K$, so the lemma is obvious. So we assume the lemma holds for all vector spaces W with $\dim W < \dim V$.

Now we consider two cases. (1) S contains a nonscalar semisimple element A, and its negation (2) S does not contain a nonscalar semisimple element.

(Case 1) For $\alpha \in K$, let V_α be the eigenspace of A associated to the eigenvalue α.

$$V_\alpha = \{v \in V : Av = \alpha v\}$$

Since A is semisimple, V is the direct sum of eigenspaces for A. (Since V is finite dimensional, there are only finitely many α for which V_α is nonzero.)

$$V = \bigoplus_{\alpha \in K} V_\alpha$$

Since A is nonscalar, $\dim V_\alpha < \dim V$ for each α. Also note that for $s \in S$, and $v \in V_\alpha$, we have

$$As(v) = sA(v) = s(\alpha v) = \alpha s(v)$$

so $s(v) \in V_\alpha$, so $S(V_\alpha) \subset V_\alpha$. Thus by induction, each V_α has a basis for which S is simultaneously upper triangular, with semisimple elements diagonal. Since every endomorphism in S is a direct sum of endomorphisms of the V_α, the result also holds for V (since direct sum of endomorphisms corresponds to block diagonal matrices, direct sums of upper trianglar/diagonal matrix representations are respectively upper triangular/diagonal matrix representations).

(Case 2) If every element of S is scalar, then we are done, since every S is simulaneously diagonalized by any basis. So we may assume S contains a nonscalar element A which is NOT semisimple. Since K is algebraically closed, A has an eigenvalue α.

Let V_α be the associated eigenspace. Since A is nonscalar, $\dim V_\alpha < \dim V$. As in case (1), $S(V_\alpha) \subset V_\alpha$, so each $s \in S$ induces an endomorphism of the quotient space V/V_α.

By induction hypotethesis, the result holds for V_α and for V/V_α, so there is a basis $\{v_1, \ldots, v_k\}$ of V_α such that every element of S (acting on V_α) is upper triangular and the semisimple elements are diagonal, and there is a basis $\{\overline{v}_{k+1}, \ldots, \overline{v}_n\}$ of V/V_α where S is simultaneously upper triangular, and semisimple elements are diagonal.

By construction of these bases, for each $s \in S$, and each j such that $k+1 \leq j \leq n$, we have $s(\overline{v}_j) \in \operatorname{span}\{\overline{v}_{k+1}, \ldots, \overline{v}_j\}$. Let $v_{k+1}, \ldots, v_n$ be lifts of $\overline{v}_{k+1}, \ldots, \overline{v}_n$. Then $\{v_1, \ldots, v_n\}$ is a basis of V, and $s(v_j) \in \operatorname{span}\{v_1, \ldots, v_j\}$. So with respect to this basis, S is simultaneously upper triangular.

It remains to show that semisimple elements of S are diagonal in this basis, which is to say, $v_1, \ldots, v_n$ eigenvectors for any $s \in S$ which is semisimple. By construction of $v_1, \ldots, v_k$, they are eigenvectors of s, since s being semisimple means it is diagonal in the basis $\{v_1, \ldots, v_k\}$ for V_α. Also, for $k+1 \leq j \leq n$, $\overline{v}_j$ is an eigenvector of s, which is to say, $s(\overline{v}_j) = \lambda_j \overline{v}_j$ for some $\lambda_j \in K$, so $s(v_j) = \lambda_j v_j + w_j$ for some $w_j \in V_\alpha$.

How to show $w_j = 0$? It's not so easy to see that this is possible, even with modifying the choice of lift v_j for $\overline{v}_j$, since you have to do it for all s at the same time. I don't know how to finish the proof. □

Corollary 4.6. *Let V be a finite dimensional vector space over K, and let $S \subset \operatorname{End}(V)$ be a set of semisimple elements. Then there is a basis of V which simultaneously diagonalizes all elements of S if and only if S pairwise commutes.*

Proof. If everything can be simultaneously diagonalized, then clearly they all pairwise commute. The converse is the content of the previous lemma. □

4.2 Jordan decomposition over algebraically closed fields

As before, K is a fixed algebraically closed field.

Definition 4.7. Let $X \in \mathrm{End}(V)$, and let $\alpha \in K$. The **generalized eigenspace** of X associated to α is

$$V^\alpha = \{v \in V : (X-\alpha)^r v = 0 \,\text{for some}\; r \geq 0\}$$

Lemma 4.8 (Generalized eigenspace decomposition). *Let V be a finite dimensional vector space over K, and let $X \in \mathrm{End}(V)$, and let*

$$f(T) = \prod_{i=1}^{m} (T-\alpha_i)^{n_i}$$

be the minimal polynomial of X. Then

$$V = \bigoplus_{i=1}^{m} V^{\alpha_i}$$

and $V^\alpha = \ker(X-\alpha_i)^{n_i}$ and the minimal polynomial of $X|_{V^{\alpha_i}} \in \mathrm{End}(V^{\alpha_i})$ is $g_i(T) = (T-\alpha_i)^{n_i}$.

Proof. We omit the proof that $V = \oplus V^{\alpha_i}$, since it requires some build up of other lemmas. We prove the other two statements. It is clear that $\ker(X-\alpha_i)^{n_i} \subset V^{\alpha_i}$. Since $(X-\alpha_i)^{n_i} = 0$, the reverse inclusion is also clear. Regarding the minimal polynomial of $X|_{V^{\alpha_i}}$, it is clear that $g_i(X|_{V^{\alpha_i}}) = 0$. If a lower power of $(X-\alpha_i)$ was zero on all of V^{α_i}, then $f(T)$ would not be the minimal polynomial of X, so g_i is the minimal polynomial on V^{α_i}. □

Proposition 4.9 (Jordan decomposition). *Let V be a finite dimensional vector space over K, and let $X \in \mathrm{End}(V)$. Then there exist $S, N \in \mathrm{End}(V)$ such that*

1. $X = S + N$
2. *S is semisimple*
3. *N is nilpotent*
4. $SN = NS$

Furthermore, S, N are uniquely determined by these properties, and S, N can be written as polynomials in $K[X]$ with zero constant term.

For obvious reasons, the S, N of the proposition are called the **semisimple part** and **nilpotent part** of X.

Proof. Let

$$f(T) = \prod_{\alpha} (T-\alpha)^{n_\alpha} \in K[T]$$

be the minimal polynomial of X (each α is in K and the α are the eigenvalues of X, T is a variable). Let

$$V^\alpha = \{v \in V : (X-\alpha)^m v = 0 \text{ for some } m \geq 0\}$$

be the generalized eigenspace of X associated to α. By Lemma 4.8, α is the only eigenvalue of $X|_{V^\alpha}$. Since $V = \bigoplus V^\alpha$, we may define $S \in \operatorname{End}(V)$ by defining it on each V^α. Set

$$S|_{V^\alpha} = \alpha \operatorname{Id}|_{V^\alpha}$$

Now S is clearly semisimple, and $SX = XS$, since $X(V^\alpha) \subset V^\alpha$. Now set $N = X - S$. Clearly N commutes with S and X as well. Note that N is nilpotent on each V^α, and each V^α is an invariant subspace (of N), so N is nilpotent. Thus N, S exist and have the listed properties. It remains to show that N, S are unique and can be written as polynomials in X with zero constant term.

The polynomials $(T-\alpha)^{n_\alpha}$ are pairwise coprime, so by the Chinese Remainder Theorem, there exists a polynomial $p(T) \in K[T]$ such that

$$\begin{aligned} &p(T) \equiv \alpha \bmod (T-\alpha)^{n_\alpha} \\ &p(T) \equiv 0 \bmod T \qquad\qquad \text{if } V^0 = 0 \end{aligned}$$

By Lemma 4.8, on V^α, $(X-\alpha)^{n_\alpha} = 0$, hence

$$p\left(X|_{V^\alpha}\right) = \alpha = S|_{V^\alpha}$$

Since V decomposes into the V^α, this shows $p(X) = S$. So S is a polynomial in X, and $p(X) = 0 \bmod X$, so the constant term is zero. Since $N = X - S = X - p(X)$, N is also a polynomial in X with no constant term.

Finally, we prove uniqueness. Let $X = S + N$ and $X = S' + N'$ be two such decompositions. S, S', N, N' are polynomials in X, so everything commutes, so we may find a basis of V which simultaneously makes them all upper triangular, and makes S, S' diagonal. We have $S - S' = N - N'$, and from the constructed basis it is clear that $S - S'$ is semisimple because it is diagonal, and $N - N'$ is nilpotent, since N, N' are nilpotent (use binomial theorem or something), so $S - S' = N - N'$ is both semisimple and nilpotent, so it is zero. Hence $S = S'$ and $N = N'$, proving uniqueness. □

Proposition 4.10. *Let V be a finite dimensional K-vector space and let $X \in \operatorname{End}(V)$, and let $X = S + N$ be the Jordan decomposition. If X is invertible (e.g. if $X \in \operatorname{Aut}(V)$) then S is also invertible, and S^{-1} is a polynomial in X.*

Proof. Let

$$f(T) = \prod_\alpha (T-\alpha)^{n_\alpha}$$

be the minimal polynomial of X, and let $p(T) \in K[T]$ be the polynomial such that $p(X) = S$. From the construction of $p(T)$ in the proof of the previous proposition, we had

$$p(T) \equiv \alpha \bmod (T-\alpha)^{n_\alpha}$$

Also, note that
$$f(T) \equiv 0 \bmod (T-\alpha)^{n_\alpha}$$
so $p(T)$ and $f(T)$ are coprime. Since $K[T]$ is a Euclidean domain, there exist $q(T), g(T) \in K[T]$ such that
$$1 = pq + fg = p(T)q(T) + f(T)g(T)$$
Plugging in X, the fg term vanishes because $f(X) = 0$, so

$$1 = p(X)q(X) = Sq(X)$$

Hence S is invertible and $S^{-1} = q(X)$ is a polynomial in X. □

Proposition 4.11 (Multiplicative Jordan decomposition)**.** *Let V be a finite dimensional K-vector space, and let $X \in \mathrm{Aut}(V)$. Then there exist $S, U \in \mathrm{Aut}(V)$ such that*

1. $X = SU$
2. *S is semisimple*
3. *U is unipotent*
4. $SU = US$

Furthremore, S, U are uniquely determined by these properties, and S, U can be written as polynomials in $K[X]$, with S having no constant term.

A slicker way to phrase the proposition is this: any automorphism of a finite dimesional vector space can be uniquely written as a product of a semisimple element and a nilpotent element that commute with each other.

For obvious reasons, the S of the proposition is called the **semisimple part** of X, and the U is called the **unipotent part** of X.

Proof. Let $X \in \mathrm{Aut}(V)$. We can write X uniquely as $X = S + N$ with S semisimple and N nilpotent. By Proposition 4.10, S is invertible, so we can write X as

$$X = S + N = S(\mathrm{Id} + S^{-1}N)$$

Since N is nilpotent and S^{-1}, N commute, $S^{-1}N$ is nilpotent, so $U = \mathrm{Id} + S^{-1}N$ is unipotent. It is clear that $SU = US$, and by Proposition 4.10, S^{-1} is a polynomial in X, so U is as well. It just remains to show uniqueness.

Suppose $X = SU = S'U'$ with S, S' semisimple and U, U' unipotent. Let $N = SU - \mathrm{Id}$ and $N' = S'U' - \mathrm{Id}$, so $U = \mathrm{Id} + S^{-1}N$. Since U is unipotent, $U - \mathrm{Id} = S^{-1}N$ is nilpotent. Since S^{-1} is invertible, N must be nilpotent. Similarly, N' is nilpotent. Now observe

$$X = S + N = S + SU - \mathrm{Id} = S - \mathrm{Id} + S'U' = S - \mathrm{Id} + N' + \mathrm{Id} = S' + N'$$

By uniqueness of the (additive) Jordan decomposition, $S = S', N = N'$, so $U = U'$. □

4.3 Jordan decomposition over perfect fields

Algebraic closure of K is not entirely necessary for Jordan decomposition. A weaker but sufficient condition for the unique decomposition $X = S + N$ is that K is a perfect field, which means that every algebraic extension of K is separable. In this section, we show that Jordan decomposition still works in the perfect case..

For this section, assume K is a non-algebraically closed field, and let $\overline{K}$ denote the algebraic closure. Let V be a finite dimensional K-vector space. The first issue we would like to address is that our previous equivalent definitions of semisimplicity for an endomorphism $X \in \text{End}_K(V)$ are no longer equivalent. Before we can do that, we need some other things.

Definition 4.12. Let V be a finite dimensional K-vector space. Associated to V is the $\overline{K}$-vector space

$$V_{\overline{K}} = V \otimes_K \overline{K}$$

The $\overline{K}$ action on V_K is done in the expected way: for $\lambda, \alpha \in \overline{K}, v \in V$, λ acts on the simple tensor $v \otimes \alpha$ via

$$\overline{K} \times V_{\overline{K}} \to V_{\overline{K}} \qquad \lambda \cdot (v \otimes \alpha) = v \otimes (\lambda\alpha)$$

Since the simple tensors give a basis of V_K, extending this by linearity gives a $\overline{K}$ action.

Definition 4.13. Similar to the above, if $\phi : V \to W$ is a morphism of K-vector spaces (a K-linear map), then there is an associated $\overline{K}$-linear map

$$\phi_{\overline{K}} = \phi \otimes \text{Id} : V_{\overline{K}} \to W_{\overline{K}} \qquad v \otimes \alpha \mapsto \phi(v) \otimes \alpha$$

As before, this only defines $\phi_{\overline{K}}$ for simple tensors, but we extend by linearity. In particular, if $X \in \text{End}_K(V)$, then the associated map $X_{\overline{K}}$ lies in $\text{End}_{\overline{K}}(V_{\overline{K}})$. Even better, this is an embedding

$$\text{End}_K(V) \to \text{End}_{\overline{K}}(V_{\overline{K}}) \qquad \phi \mapsto \phi_{\overline{K}}$$

Definition 4.14. Let $\overline{\phi} \in \text{End}_{\overline{K}}(V_{\overline{K}})$. If there exists $\phi \in \text{End}_K(V)$ such that $\overline{\phi} = \phi_{\overline{K}}$, then we say $\overline{\phi}$ **comes from** ϕ, or we may just say $\overline{\phi}$ comes from V, or that ϕ comes from a K-endomorphism of V.

Remark 4.15. Much of the difficulty in extending Jordan decompositions for non-algebraically closed fields lies in the question of determining when a given $\overline{K}$-linear endomorphism of $V_{\overline{K}}$ comes from a K-linear endomorphism of V. In general, something in $\text{End}_{\overline{K}}(V_{\overline{K}})$ does not usually come from V, but when things work out for Jordan decompositions, it is because we were able to show something came from V.

Definition 4.16. $X \in \text{End}_K(V)$ is **semisimple** if $X_{\overline{K}}$ is semisimple. Note that this is equivalent to the minimal polynomial of X being square-free, or that every X-invariant subspace of V has a complementary X-invariant subspace.

Even though the full power of Jordan decomposition does not hold in the case of a general non-algebraically closed field, the above suggests a way forward. Since $\overline{K}$ is algebraically closed, $X_{\overline{K}}$ has a Jordan decomposition

$$X_{\overline{K}} = \overline{S} + \overline{N}$$

The question is then, do $S_{\overline{K}}$ and $N_{\overline{K}}$ come from endomorphisms $S, N \in \text{End}_K(V)$? There is no reason to think so.

Definition 4.17. Let V, K be as above, and suppose K is perfect, so that $\overline{K} = K^{\text{sep}}$, which means that $\overline{K}/K$ is a Galois extension. Let $G = \text{Gal}(\overline{K}/K)$. For $\sigma \in G$, and $\overline{Y} \in \text{End}_{\overline{K}}(V_{\overline{K}})$, define

$$\overline{Y}^\sigma \in \text{End}_{\overline{K}}(V_{\overline{K}})$$

as the composition

$$V_{\overline{K}} \xrightarrow{\text{Id} \otimes \sigma^{-1}} V_{\overline{K}} \xrightarrow{\overline{Y}} V_{\overline{K}} \xrightarrow{\text{Id} \otimes \sigma} V_{\overline{K}}$$

If e is the identity of G, it is clear that $\overline{Y}^e = \overline{Y}$, and it also clear that for $\sigma, \tau \in G$,

$$\left(\overline{Y}^\sigma\right)^\tau = \overline{Y}^{(\tau\sigma)}$$

Thus we have a group action

$$G \times \text{End}_{\overline{K}}(V_{\overline{K}}) \to \text{End}_{\overline{K}}(V_{\overline{K}}) \qquad (\sigma, \overline{Y}) \mapsto \overline{Y}^\sigma$$

Note that in this action, each $\sigma \in G$ acts $\overline{K}$-linearly on $\text{End}_{\overline{K}}(V_{\overline{K}})$.

Remark 4.18. Let $\sigma \in G = \text{Gal}(\overline{K}/K)$ as in the previous definition. If $\overline{Y} \in \text{End}_{\overline{K}}(V_{\overline{K}})$ is semisimple, then $\overline{Y}^\sigma$ is also semisimple, and the analogous statements for nilpotent and unipotent hold as well.

Remark 4.19. Let V, K, G be as in the previous definition. If $X \in \text{End}_K(V)$, then $X_{\overline{K}}$ is fixed under the action of G. That is, for $\sigma \in G$,

$$X_{\overline{K}}^\sigma = X_{\overline{K}}$$

Why? Remember that $X_{\overline{K}}^\sigma$ is defined as the composition

$$X_{\overline{K}}^\sigma : V \otimes_K \overline{K} \xrightarrow{\text{Id} \otimes \sigma^{-1}} V \otimes_K \overline{K} \xrightarrow{X_{\overline{K}} = X \otimes \text{Id}} V \otimes_K \overline{K} \xrightarrow{\text{Id} \otimes \sigma} V \otimes_K \overline{K}$$

The middle part $X_{\overline{K}}$ acts only on V, so the actions of σ^{-1} and σ cancel out.

The next goal is to obtain a sort of converse to this remark. That is, we want to say that anything satisfying $\overline{X}^\sigma = \overline{X}$ for every $\sigma \in G$ comes from an endomorphism of V. However, this statement is not actually true, so we need to refine it by adding additional hypotheses in the form of restrictions on X. First, we give a long and concrete example using all of these definitions.

Example 4.20. Let $K = \mathbb{R}$ (which is perfect), $\overline{K} = \mathbb{C}, V = \mathbb{R}^2, V_{\overline{K}} = \mathbb{R}^2 \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}^2, G = \text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the canonical basis of $V = \mathbb{R}^2$, and fix the basis $e_1 \otimes 1 + e_2 \otimes i, e_1 \otimes 1 - e_2 \otimes i$ for $V_{\overline{K}} = V \otimes_{\mathbb{C}} \mathbb{C}$. Fix $\theta \in \mathbb{R}$, and let $\overline{S} \in \text{End}_{\overline{K}}(V_{\overline{K}}) \cong \text{GL}_2(\mathbb{C})$ be the automorphism represented by the matrix

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

with respect to the aforementioned basis of $V_{\overline{K}}$. That is,

$$\overline{S}(e_1 \otimes 1 + e_2 \otimes i) = e^{i\theta}(e_1 \otimes 1 + e_2 \otimes i) = e_1 \otimes e^{i\theta} + e_2 \otimes ie^{i\theta}$$
$$\overline{S}(e_1 \otimes 1 - e_2 \otimes i) = e^{-i\theta}(e_1 \otimes 1 - e_2 \otimes i) = e_1 \otimes e^{-i\theta} - e_2 \otimes ie^{-i\theta}$$

Note that $\overline{S}$ is semisimple, since it is diagonal in this basis. The eigenvalues are $e^{i\theta}, e^{-i\theta}$, which are notably not in $\mathbb{R}$, unless θ is an integer multiple of π.

Now consider the Galois action of G on $\mathrm{End}_{\overline{K}}(V_{\overline{k}})$. There is only one nontrivial element, which is complex conjugation, and we denote by σ. We can compute $\overline{S}^{\sigma}$ acting on our basis vectors explicitly as follows.

$$\begin{aligned}
\overline{S}^{\sigma}(e_1 \otimes 1 + e_2 \otimes i) &= (1 \otimes \sigma)\overline{S}(e_1 \otimes 1 - e_2 \otimes i) \\
&= \sigma\left(e^{-i\theta}(e_1 \otimes 1 - e_2 \otimes i)\right) \\
&= e^{i\theta}(e_1 \otimes 1 + e_2 \otimes 1) \\
&= \overline{S}(e_1 \otimes 1 + e_2 \otimes i) \\
\overline{S}^{\sigma}(e_1 \otimes 1 - e_2 \otimes i) &= (1 \otimes \sigma)\overline{S}(e_1 \otimes 1 + e_2 \otimes i) \\
&= (1 \otimes \sigma)(e^{i\theta}(e_1 \otimes 1 + e_2 \otimes i) \\
&= e^{-i\theta}(e_1 \otimes 1 - e_2 \otimes i) \\
&= \overline{S}(e_1 \otimes 1 - e_2 \otimes i)
\end{aligned}$$

Thus $\overline{S}^{\sigma} = \overline{S}$. As we mentioned before, we want this to somehow tell us that $\overline{S}$ comes from V, perhaps with additional hypotheses. However, in the case of this example, we can work out that $\overline{S}$ comes from V more directly, which we will do in a moment.

Setting this aside, we now calculate the matrix of $\overline{S}$ with respect to the more usual basis $e_1 \otimes 1, e_2 \otimes 1$ of $V_{\overline{K}}$ as follows. Adding the two previous equations for $\overline{S}$ acting on the original basis, we obtain

$$\begin{aligned}
2\overline{S}(e_1 \otimes 1) &= e_1 \otimes e^{i\theta} + e_2 \otimes ie^{i\theta} + e_1 \otimes e^{-i\theta} - e_2 \otimes ie^{-i\theta} \\
&= e_1 \otimes \left(e^{i\theta} + e^{-i\theta}\right) + e_2 \otimes \left(ie^{i\theta} - ie^{-i\theta}\right) \\
&= e_1 \otimes (2\cos\theta) - e_2 \otimes (2\sin\theta) \\
\overline{S}(e_1 \otimes 1) &= (\cos\theta)(e_1 \otimes 1) - (\sin\theta)(e_2 \otimes 1)
\end{aligned}$$

Similarly, subtracting the equations gives

$$\begin{aligned}
2\overline{S}(e_2 \otimes i) &= e_1 \otimes e^{i\theta} + e_2 \otimes ie^{i\theta} - e_1 \otimes e^{-i\theta} + e_2 \otimes ie^{-i\theta} \\
&= e_1 \otimes \left(e^{i\theta} - e^{-i\theta}\right) + e_2 \otimes \left(ie^{i\theta} + ie^{-i\theta}\right) \\
&= e_1 \otimes (2i\sin(\theta)) + e_2 \otimes (2i\cos(\theta)) \\
\overline{S}(e_2 \otimes 1) &= (\sin\theta)(e_1 \otimes 1) + (\cos\theta)(e_2 \otimes 1)
\end{aligned}$$

Thus with respect to the basis $e_1 \otimes 1, e_2 \otimes 1$, the matrix of $\overline{S}$ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Note that this is the usual matrix which represents rotation by the angle θ when acting on $v \in \mathbb{R}^2$. It is now clear that $\overline{S}$ comes from this endomorphism of $V = \mathbb{R}^2$, that is, the linear map represented by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with respect to the standard basis e_1, e_2 for $\mathbb{R}^2$. This concludes the example.

Now after just one more short definition, we can get to our key statement for recognizing when something in $\mathrm{End}_{\overline{K}}(V_{\overline{K}})$ comes from $\mathrm{End}_K(V)$.

Definition 4.21. Let G be a group acting on a set A. The set of **fixed points** is $A^G = \{a \in A : \sigma a = a, \forall \sigma \in G\}$.

Lemma 4.22. *Let L/K be a finite Galois extension, and set $G = \mathrm{Gal}(L/K)$. Consider the group action*

$$G \times M_n(L) \to M_n(L) \qquad (\sigma, A) \mapsto A^\sigma = \sigma \circ A \circ \sigma^{-1}$$

Then $M_n(L)^G = M_n(K)$. That is, for $A \in M_n(L)$, $A \in M_n(K)$ if and only if $A = A^\sigma$ for all $\sigma \in G$.

Proof. It is clear that if $A \in M_n(K)$, then $A^\sigma = A$, so $M_n(K) \subset M_n(L)^G$. For the converse, consider $v \in L^n$, then

$$A^\sigma(v) = \sigma(A(\sigma^{-1}(v))) = \sigma(A)\sigma(\sigma^{-1}(v)) = \sigma(A)v$$

where $\sigma(A)$ refers to applying σ to each entry of A. Thus $A^\sigma = \sigma(A)$. So if $A^\sigma = A$, then all entries of A are fixed by G, and $A \in M_n(K)$. Hence $M_n(L)^G \subset M_n(K)$, so they are equal. □

Lemma 4.23. *Let V, K, G be as above, in particular K is perfect. If $\overline{T} \in \mathrm{End}_{\overline{K}}(V_{\overline{K}})^G$, then there exists $T \in \mathrm{End}_K(V)$ such that $\overline{T} = T_{\overline{K}}$. That is, if $\overline{T}^\sigma = \overline{T}$ for every $\sigma \in G$, then $\overline{T}$ comes from a K-endomorphism of V.*

Proof. This is basically just a "coordinate-free" version of the previous lemma. Roughly speaking, all we have to do is fix a basis of V and this will follow, but we include the details. Fix a K-basis $\beta = \{v_1, \ldots, v_n\}$ of V, which induces a $\overline{K}$-basis $\overline{\beta} = \{v_1 \otimes 1, \ldots, v_n \otimes 1\}$ of $V_{\overline{K}}$. This gives isomorphisms

$$\begin{aligned} V^n &\cong K^n & v_i &\mapsto e_i \\ \mathrm{End}_K(V) &\cong M_n(K) & T &\mapsto [T]_\beta \\ V_{\overline{K}}^n &\cong \overline{K}^n & v_i \otimes 1 &\mapsto e_i \\ \mathrm{End}_{\overline{K}}(V_{\overline{K}}) &\cong M_n\left(\overline{K}\right) & \overline{T} &\mapsto \left[\overline{T}\right]_{\overline{\beta}} \end{aligned}$$

where e_i is the column vector with 1 in the ith place and zeroes elsewhere, and $[T]_\beta$ is the matrix of T in the basis β, and $\left[\overline{T}\right]_{\overline{\beta}}$ is the matrix of $\overline{T}$ in the basis $\overline{\beta}$. The notation is chosen so that

$$[Tv_i]_\beta = [T]_\beta [v_i]_\beta = [T]_\beta e_i$$

Also note that under the isomorphism $\mathrm{End}_{\overline{K}}(V_{\overline{K}}) \cong M_n(\overline{K})$, the K-subspaces $\mathrm{End}_K(V)$ and $M_n(K)$ correspond, via the isomorphism $\mathrm{End}_K(V) \cong M_n(K)$. We can depict this with the following commutative diagram.

$$\begin{array}{ccc} \mathrm{End}_K(V) & \xrightarrow{\cong} & M_n(K) \\ {\scriptstyle T \mapsto T_{\overline{K}}} \big\downarrow & & \big\downarrow \\ \mathrm{End}_{\overline{K}}(V_{\overline{K}}) & \xrightarrow{\cong} & M_n(\overline{K}) \end{array}$$

Our isomorphisms are respectively isomorphisms of K-algebras and $\overline{K}$-algebras, but we claim that the second one is also an isomorphism of G-modules. Let $\overline{T} \in \mathrm{End}_{\overline{K}}(V_{\overline{K}})$ and $\sigma \in G$. Then

$$\begin{aligned} \left[\overline{T}^{\sigma}\right]_{\overline{\beta}}(e_i) &= \left[(1 \otimes \sigma)\overline{T}(1 \otimes \sigma^{-1})\right]_{\overline{\beta}}(e_i) \\ &= \left[(1 \otimes \sigma)\overline{T}(1 \otimes \sigma^{-1})(v_i \otimes 1)\right]_{\overline{\beta}} \\ &= \left[(1 \otimes \sigma)\overline{T}(v_i \otimes 1)\right]_{\overline{\beta}} \\ &= \sigma\left[\overline{T}(v_i \otimes 1)\right]_{\overline{\beta}} \\ &= \sigma\left[\overline{T}\right]_{\overline{\beta}}(e_i) \\ &= \sigma\left[\overline{T}\right]_{\overline{\beta}}\sigma^{-1}(e_i) \\ &= \left[\overline{T}\right]_{\overline{\beta}}^{\sigma}(e_i) \end{aligned}$$

Since they agree on basis, this shows $\left[\overline{T}^{\sigma}\right]_{\overline{\beta}} = \left[\overline{T}\right]_{\overline{\beta}}^{\sigma}$, hence it is a morphism of G-modules as claimed. Now let $\overline{T} \in \mathrm{End}_{\overline{K}}(V_{\overline{K}})^G$. Since $\left[\overline{T}\right]_{\overline{\beta}}$ has only fintiely many entries, there is a finite Galois extension L/K such that $\left[\overline{T}\right]_{\overline{\beta}} \in M_n(L)$. Let $\sigma \in G$, so we have $\overline{T} = \overline{T}^{\sigma}$, so

$$\left[\overline{T}\right]_{\overline{\beta}} = \left[\overline{T}^{\sigma}\right]_{\overline{\beta}} = \left[\overline{T}\right]_{\overline{\beta}}^{\sigma}$$

Since $\sigma \in G$ was arbitrary, it also holds for $\sigma|_L \in \mathrm{Gal}(L/K)$. Then by the previous lemma, $\left[\overline{T}\right]_{\overline{\beta}} \in M_n(K)$. Then since $M_n(K)$ corresponds to the image of $\mathrm{End}_K(V)$ in $\mathrm{End}_{\overline{K}}(V_{\overline{K}})$, $\overline{T} = T_{\overline{K}}$ for some $T \in \mathrm{End}_K(V)$. □

Proposition 4.24 (Jordan decomposition for perfect fields)**.** *Let V, K be as above. Let $X \in \mathrm{End}_K(V)$ and*

$$X_{\overline{K}} = \overline{S} + \overline{N}$$

be the Jordan decomposition of $X_{\overline{K}}$. If K is perfect, then there are unique $S, N \in \mathrm{End}_K(V)$ satisfying

$$\overline{S} = S_{\overline{K}} \qquad \overline{N} = N_{\overline{K}} \qquad X = S + N$$

with S is semisimple and N is nilpotent, and $SN = NS$. Also, S, N are polynomials in X with no constant term.

Proof. As K is perfect, $\overline{K}/K$ is Galois. Let $G = \mathrm{Gal}(\overline{K}/K)$ and consider the $\overline{K}$-linear G-action on $\mathrm{End}_{\overline{K}}(V_{\overline{K}})$ defined previously. Let $\sigma \in G$. Applying the action to $X_{\overline{K}} = \overline{S} + \overline{N}$, and using Remark 4.19, we obtain

$$X_{\overline{K}}^{\sigma} = \overline{S}^{\sigma} + \overline{N}^{\sigma} = X_{\overline{K}}^{\sigma} = X_{\overline{K}} = \overline{S} + \overline{N}$$

By Remark 4.18, $\overline{S}^{\sigma}$ is semisimple and $\overline{N}^{\sigma}$ is nilpotent. Thus by uniqueness of Jordan decomposition, $\overline{S} = \overline{S}^{\sigma}$ and $\overline{N} = \overline{N}^{\sigma}$. Since $\sigma \in G$ was arbitrary, by Lemma 4.23, $\overline{S}, \overline{N}$ come from V, which is to say, there exist $S, N \in \mathrm{End}_K(V)$ such that $\overline{S} = S_{\overline{K}}$ and $N_{\overline{K}} = \overline{N}$.

It is immediate that S is semisimple, and that N is nilpotent, and that $X = S + N$. The commutativity $SN = NS$ follows from $S_{\overline{K}} N_{\overline{K}} = N_{\overline{K}} S_{\overline{K}}$, and uniqueness of S, N follow from uniqueness of $\overline{S}, \overline{N}$. The statement about polynomials in X follows from the fact that $\overline{S}, \overline{N}$ are polynomials in $X_{\overline{K}}$. □

4.4 Failure of Jordan decomposition over imperfect fields

In this section we give an example of particular matrix over an imperfect field which Jordan decomposition fails. Let $\mathbb{F}_q$ be the field with q elements. The canonical example of an imperfect field is $\mathbb{F}_q(T)$, where T is a transcendant variable. For concreteness, we will consider $\mathbb{F}_2(T)$, though in principal this example could be carried out more generally for $\mathbb{F}_q(T)$.

Let $K = \mathbb{F}_2(T)$ and let V be a two dimensional K-vector space, which we identify with column vectors. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the canonical basis for V. Given any basis of V, we get an isomorphism (of K-algebras) between $\mathrm{GL}_2(K)$ and $\mathrm{End}_K(V)$. Let $X \in \mathrm{End}_K(V)$ be the endomorphism corresponding to the following matrix, with respect to the canonical basis.

$$X = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix} \in \mathrm{End}_K(V)$$

As in the previous section, we may consider X as an endomorphism of $V_{\overline{K}} = V \otimes_K \overline{K}$, by considering $X_{\overline{K}} = X \otimes \mathrm{Id} \in \mathrm{End}_{\overline{K}}(V_{\overline{K}})$. With respect to the canonical basis of $V_{\overline{K}}$ (which is two dimensional over $\overline{K}$), $X_{\overline{K}}$ has the same matrix as X. We wish to explicitly find the Jordan decomposition of $X_{\overline{K}}$. One way to do this is to find a basis of $V_{\overline{K}}$ in which $X_{\overline{K}}$ is upper triangular.

We omit the details of finding this basis at the moment, and just present it as given. Consider the basis $\left\{e_1 + \sqrt{T} e_2, e_2\right\}$ of $V_{\overline{K}}$. Changing to this basis corresponds to conjugating by the matrix $A = \begin{pmatrix} 1 & 0 \\ \sqrt{T} & 1 \end{pmatrix}$, so we conjugate X by this matrix. Note that $A = A^{-1}$, keeping in mind that $2 = 0$.

$$AXA^{-1} = \begin{pmatrix} 1 & 0 \\ \sqrt{T} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{T} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ T & \sqrt{T} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sqrt{T} & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{T} & 1 \\ 2T & \sqrt{T} \end{pmatrix} = \begin{pmatrix} \sqrt{T} & 1 \\ 0 & \sqrt{T} \end{pmatrix}$$

So in the basis $\left\{e_1 + \sqrt{T} e_2, e_2\right\}$ for $V_{\overline{K}}$, $X_{\overline{K}}$ is represented by the matrix

$$\begin{pmatrix} \sqrt{T} & 1 \\ 0 & \sqrt{T} \end{pmatrix} = \begin{pmatrix} \sqrt{T} & 0 \\ 0 & \sqrt{T} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \overline{S} + \overline{N}$$

The above gives a Jordan decomposition for $X_{\overline{K}}$. Now the question is, does this decomposition descend to a decomposition of X? That is, are there $S, N \in \mathrm{End}_K(V)$ such that $\overline{S} = S_{\overline{K}}$ and $\overline{N} = N_{\overline{K}}$? For N this is fine, N clearly comes from the same matrix over K. However, $\overline{S}$ involves the entry $\sqrt{T}$, which does not lie in K. This strongly suggests that $\overline{S}$ does not come from V.

Consider the set $V' = \{v \otimes 1 : v \in V\} \subset V_{\overline{K}}$. Any endomorphism coming from $\mathrm{End}_K(V)$ keeps V' invariant. That is, if $Y \in \mathrm{End}_K(V)$, then Y restricts to a map $V' \to V'$. Thus, because $\overline{S}$ does not keep V' invariant, $\overline{S}$ does not come from V. Hence the Jordan decomposition for $X_{\overline{K}}$ does not descend to a Jordan decomposition over K.

4.5 Jordan decomposition in infinite dimensions

We return to assuming our field K to be algebraically closed. In the previous sections, we have only worked with vector spaces of finite dimension over K. We have successfully worked out the details of Jordan decomposition in this case.

However, even when working with finite dimensional algebraic groups, the associated coordinate K-algebra is almost never finite dimensional. It is usually a quotient or localization of a polynomial ring $K[x_1, \ldots, x_n]$, and even the polynomial ring in one variable $K[x]$, while being finitely generated as an algebra, is infinite dimensional as a K-vector space.

All this to say, it will occasionally be useful/necessary for us to have Jordan decompositions in this context. To my knowledge, Jordan decompositions do not work in general for endomorphisms of infinite dimensional vector spaces. But as we show in this section, they do work out for "locally finite" endomorphisms.
Throughout this section, let V be a vector space over K, not necessarily finite dimensional. (For all of these results, if V is finite dimensional then we already know the result, so for the purposes of reading you may as well assume V is infinite dimensional over K.)

Definition 4.25. An endomorphism $T \in \mathrm{End}(V)$ is **locally finite** if V can be written as $V = \sum V_\lambda$, where each V_λ is a finite dimensional T-invariant subspace.

Remark 4.26. Possibly utilizing Zorn's lemma, being locally finite is equivalent to being able to write V as a *direct* sum of finite dimensional invariant subspaces, $V = \bigoplus_\lambda V_\lambda$.

Lemma 4.27. *$T \in \mathrm{End}(V)$ is locally finite if and only if each $v \in V$ is contained in a finite dimensional T-invariant subspace.*

Proof. If T is locally finite, then $V = \sum V_\lambda$, and each $v \in V$ is contained in some finite sub-sum of the V_λ. Conversely, if each $v \in V$ is contained in a finite dimensional T-invariant subspace, then T is the sum of those subspaces. □

Remember that if V is infinite dimensional, then $T \in \mathrm{End}(V)$ may be injective but not surjective (if V is finite dimensional, then injectivity implies surjectivity by dimension counting). Despite this, the next lemma shows that for locally finite endomorphisms, this property does carry over from the finite case.

Lemma 4.28. *Let $T \in \mathrm{End}(V)$ be locally finite. Then the following are equivalent.*

1. *T is injective.*

2. *T is invertible.*

3. *All eigenvalues of T are nonzero.*

Proof. The equivalence (2) $\iff$ (3) is clear, as is (1) $\implies$ (2). All that remains is (2) $\implies$ (1). Let $v \in V$. Then v is contained in a T-invariant subspace W of finite dimension, so $T : W \to W$ is a linear map, and $T|_W$ is injective. Thus $T|_W$ is surjective, so $v \in \operatorname{im} T$. Hence $T : V \to V$ is surjective. □

Remark 4.29. From the previous lemma, it is clear that if T is locally finite and invertible, then T^{-1} is also locally finite.

Proposition 4.30. *Let $T \in \operatorname{End}(V)$ be locally finite. Then there exist $S, N \in \operatorname{End}(V)$ such that*

1. *$T = S + N$*

2. *$SN = NS$*

3. *If $W \subset V$ is a finite dimensional T-invariant subspace, then S, N also keep W invariant, and $T|_W = S|_W + N|_W$ is the Jordan decomposition of $T|_W$.*

4. *S, N are locally finite.*

5. *S is semisimple (meaning that V has a basis of eigenvectors for S).*

6. *N is locally nilpotent (meaning N is nilpotent on each finite dimensional invariant subspace).*

Furthermore, S, N are uniquely determined by these properties.

Proof. Let $\{V_\lambda\}$ be the set of all finite dimensional T-invariant subspaces of V. Because T is locally finite, $V = \sum V_\lambda$. For each λ, we have a Jordan decomposition

$$T_\lambda = T|_{V_\lambda} = S_\lambda + N_\lambda$$

Define $S, N \in \operatorname{End}(V)$ by $S|_{V_\lambda} = S_\lambda$ and $N|_{V_\lambda} = N_\lambda$. To verify that this is well defined, we need to check that for two T-invariant subspaces V_λ, V_μ, the definitions of S, N agree on the intersection $W = V_\lambda \cap V_\mu$. Now observe that

$$S_\lambda|_W + N_\lambda|_W = T_\lambda|_W = T_\mu|_W = S_\mu|_W + N_\mu|_W$$

By definition, these are both Jordan decompositions for $T|_W$, so by uniqueness, $S_\lambda|_W = S_\mu|_W$ and $N_\lambda|_W = N_\mu|_W$. Thus the definition of S, N makes sense. It is now clear that properties (1) and (2) hold "locally," that is, on every finite dimensional T-invariant subspace, and such subspaces span V, so they hold everywhere.

Property (4) is clear from the construction of S and N. Property (5) follows because S is semisimple on each V_λ, meaning each V_λ has a basis of eigenvectors of S. So there is a spanning set for V of eigenvectors for S, which we can reduce if necessary to obtain a basis. Property (6) is immediate. □

Corollary 4.31. *Let $T \in \operatorname{End}(V)$ be a locally finite automorphism. Then there exist K-linear automorphisms $S, U \in \operatorname{Aut}(V)$ such that*

1. *$T = SU$*
2. *$SU = US$*
3. *S is semisimple and U is unipotent.*
4. *For each finite dimensional subspace W which is T-invariant, W is also $S-$ and U-invariant, and $T|_W = S_|WU|_W$ is the Jordan decomposition of $T|_W$.*

Furthermore, S, U are uniquely determined by theses properties.

Proof. Basically the same construction as the previous proof. Define S, U locally, and make sure that it works on intersections. □

Remark 4.32. In the decomposition $T = SU$, U has only eigenvalues of 1, so the eigenvalues (with multiplicities) for T and S are the same.

5 Jordan decomposition for all algebraic groups

Finally after our long foray into relatively pure linear algebra, we return to algebraic groups to apply the tool of Jordan decomposition that we have developed. The goal is to develop a similar type of decomposition for a general algebraic group. It is not terribly surprising that this works, since every algebraic group embeds into $\mathrm{GL}(V)$, and we have Jordan decomposition in $\mathrm{GL}(V)$.

The naive approach is the following. Embed $\phi : G \hookrightarrow \mathrm{GL}(V)$, take $x \in G$, and look at the Jordan decomposition of $\phi(x)$ in $\mathrm{GL}(V)$, $\phi(x) = SU$. Then somehow use this to obtain decomposition for x in G. The problems with this are twofold.

1. We have no reason to believe that S, U lie in the image of G under ϕ.
2. The embedding $G \hookrightarrow \mathrm{GL}(V)$ is far from unique. Suppose that S, U lie in the image of G, say $S = \phi(s), U = \phi(u)$, and $x = su$ is our proposed decomposition. If we have another embedding $\psi : G \hookrightarrow \mathrm{GL}(W)$, how do we know that $\psi(x) = \psi(s)\psi(u)$ is the Jordan decomposition of $\psi(x)$? If it is not, then our decomposition is not intrinsinc to G, but depends on the embedding, which is undesirable.

There are probably ways to handle these problems somewhat directly. However, we take a different approach to decomposing $x \in G$ into a product $x = su$ where s, u are "semisimple" and "unipotent," whatever that means for an element of an algebraic group. Our approach will have the advantage that it will be clear that it is intrinsic to G, and that under any embedding $G \hookrightarrow \mathrm{GL}(V)$, the image of $x = su$ is the Jordan decomposition in $\mathrm{GL}(V)$.

5.1 Semisimple and unipotent elements

Let (G, A) be an affine algebraic group. We may also refer to A as $K[G]$, the coordinate ring of G. Recall that for $x \in G$, we have morphisms

$$\rho_x : G \to G \qquad y \mapsto yx$$
$$\lambda_x : G \to G \qquad y \mapsto xy$$

which are isomorphisms of varieties, though not group homomorphisms. They induce K-algebra automorphisms

$$\rho_x^* : A \to A \qquad \rho_x^*(f)(y) = f(yx)$$
$$\lambda_x^* : A \to A \qquad \lambda_x^*(f)(y) = f(xy)$$

These maps are the motivation for our study of Jordan decomposition for infinite dimensional vector spaces. Recall that A is finitely generated as a K-algebra, but typically infinite dimensional as a K-vector space. However, we will see that ρ_x^*, λ_x^* are locally finite. Recall that for each $f \in A$, the subspace

$$W_f = \langle \rho_x^*(f) : x \in G \rangle_K$$

is finite dimensional (over K). Because of the relation

$$\rho_x^* \rho_y^* = \rho_{xy}^*$$

we have a group homomorphism

$$G \to \mathrm{GL}(A) \qquad x \mapsto \rho_x^*$$

Also, the relation makes it clear that W_f is ρ_x^*-invariant. In parallel,

$$\langle \lambda_x^*(f) : x \in G \rangle_K$$

is finite dimensional and λ_x^*-invariant. Summarizing, ρ_x^* and λ_x^* (for any given $x \in G$) are locally finite automorphisms of A.

Definition 5.1. Let G be an affine algebraic group. An element $s \in G$ is **semisimple** if the automorphism ρ_x^* of A is semisimple. Similarly, $u \in G$ is **unipotent** if $\rho_u^* \in \mathrm{Aut}(A)$ is unipotent.

Remark 5.2. We have apparently given preferential treatment to ρ_x^* over λ_x^* in the previous definition. The obvious reason for this is that

$$G \to \mathrm{GL}(A) \qquad x \mapsto \rho_x^*$$

is a group homomoprhism, while

$$G \to \mathrm{GL}(A) \qquad x \mapsto \lambda_x^*$$

is an "anti-homomorphism," due to the relation

$$\lambda_x^* \lambda_y^* = \lambda_{yx}^*$$

Remark 5.3. An element $s \in G$ is semisimple if and only if s^{-1} is also semisimple because

$$\rho^*_{x^{-1}} = (\rho^*_s)^{-1}$$

Similarly, $u \in G$ is unipotent if and only if u^{-1} is unipotent.

Remark 5.4. To answer the unstated question of Remark 5.2, the definitions for semisimple and unipotent group elements do not actually depend any more on ρ than λ. This is because of the relation

$$\lambda_x = i \circ \rho_{x^{-1}} \circ i$$

which induces the relation

$$\lambda^*_x = i^* \circ \rho^*_{x^{-1}} \circ i^*$$

Hence ρ^*_x is semisimple (respectively unipotent) if and only if $\rho^*_{x^{-1}}$ is semisimple (respectively unipotent) if and only if λ^*_x is semisimple (respectively unipotent). Thus the definition could have used λ^*_x instead and would be equivalent.

5.2 Main result

We start this section by stating the main result, which reads exactly the statement of Jordan decomposition for automorphisms of a vector space.

Proposition 5.5 (Jordan decomposition for algebraic groups). *Let G be an affine algebraic group and $x \in G$. Then there exist $x_s, x_u \in G$ such that*

1. $x = x_s x_u = x_u x_s$
2. *x_s is semisimple*
3. *x_u is unipotent*
4. *x_s, x_u are uniquely determined by the previous properties.*

Definition 5.6. For obvious reaons, x_s and x_u of the previous proposition are respectively called the **semisimple part** and **unipotent part** of x, and $x = x_s x_u$ is called the **Jordan decomposition** of x.

Before the proof of the main proposition, we need multiple lemmas. Recall that we assume K is an algebraically closed field.

Lemma 5.7. *Let A be a K-algebra (not necessarily commutative or even associative). Let $\sigma \in \mathrm{End}_K(A)$, $\alpha, \beta \in K$, and $f, g \in A$. Then*

$$(\sigma - \alpha\beta)^n(fg) = \sum_{i=0}^{n} \binom{n}{i} (\sigma - \alpha)^i(\alpha^{n-i} f)\sigma^i(\sigma - \beta)^{n-i}(g)$$

Proof. Then $n = 1$ case can be checked directly. Then do a notation-heavy induction step with some tricks and Pascal's identity. (See hand written notes for a weak attempt to write down most of the details.) ∎

Lemma 5.8. *Let A be a K-algebra (not necessarily commutative or even associative). Let σ be a locally finite K-algebra automorphism, and let $\sigma = su$ be the Jordan decomposition. Then s, u are K-algebra automorphisms of A.*

Proof. We already know that s, u are invertible, so if s is a K-algebra homomorphism, then $u = s^{-1}\sigma$ is as well. Hence it suffices to prove that s is a K-algebra homomorphism.

Since σ is locally finite, we can write A as a direct sum of generalized eigenspaces, which are finite dimensional.

$$A = \bigoplus_{\alpha \in K} A^\alpha$$

where

$$A^\alpha = \{f \in A | (\sigma - \alpha)^n f = 0 \text{ for some } n \geq 1\}$$

is the generalized eigenspace of σ corresponding to α. Note that S acts on A^α as multiplication by α. We claim that for $\alpha, \beta \in K$, $A^\alpha A^\beta \subset A^{\alpha\beta}$. If we prove this claim, then for $f \in A^\alpha, g \in A^\beta$,

$$s(fg) = \alpha\beta(fg) = (\alpha f)(\beta g) = s(f)S(g)$$

Then by linearity, s is a K-algebra homomorphism. So we have reduced the lemma to showing $A^\alpha A^\beta \subset A^{\alpha\beta}$. Using the formula from the previous lemma, if $f \in A^\alpha, g \in A^\beta$, then all the terms on the right hand side of the following equation are zero for n large enough.

$$(\sigma - \alpha\beta)^n (fg) = \sum_{i=0}^{n} \binom{n}{i} (\sigma - \alpha)^i (\alpha^{n-i} f) \sigma^i (\sigma - \beta)^{n-i}(g)$$

Hence $fg \in A^{\alpha\beta}$, which shows $A^\alpha A^\beta \subset A^{\alpha\beta}$ as needed. □

Lemma 5.9. *Let (G, A) be an algebraic group. If σ is a K-algebra endomorphism of A such that for every $t \in G$,*

$$\sigma \lambda_t^* = \lambda_t^* \sigma$$

then $\sigma = \rho_w^*$ for some unique $w \in G$.

Proof. Let σ be such an endomorphism. Let $t \in G$ and let $\operatorname{ev}_1 : A \to K$ be the evaluation map at the identity of G. Because G is an affine variety, the K-algebra homomorphism $\operatorname{ev}_1 \sigma : A \to K$ is evaluation at some $w \in G$. Also note that $\operatorname{ev}_1 \lambda_t^* = \operatorname{ev}_t$.

$$\operatorname{ev}_w \lambda_t^* = \operatorname{ev}_1 \sigma \lambda_t^* = \operatorname{ev}_1 \lambda_t^* \sigma = \operatorname{ev}_t \sigma$$

On the other hand,

$$\operatorname{ev}_w \lambda_t^* = \operatorname{ev}_{tw} = \operatorname{ev}_t \rho_w^*$$

Thus

$$\operatorname{ev}_t \sigma = \operatorname{ev}_t \rho_w^*$$

Since t is arbitrary, this shows $\sigma = \rho_w^*$. It just remains to show that w is unique. If $\rho_w^* = \rho_{w'}^*$, then for $f \in A$,

$$f(w) = \operatorname{ev}_1 \rho_w^*(f) = \operatorname{ev}_1 \rho_{w'}^*(f) = f(w')$$

Since f was arbitrary, every point of G is separated by some homomorphism in A, this implies $w = w'$. □

Now we can prove the main result.

Proposition 5.10 (Jordan decomposition for algebraic groups)**.** *Let G be an affine algebraic group and $x \in G$. Then there exist $x_s, x_u \in G$ such that*

1. $x = x_s x_u = x_u x_s$
2. *x_s is semisimple*
3. *x_u is unipotent*
4. *x_s, x_u are uniquely determined by the previous properties.*

Proof. Let $x \in G$, and let $\rho_x^* = SU$ be the Jordan decomposition of ρ_x^* as a K-linear automorphism of A. By Lemma 5.8, S, U are K-algebra automorphisms. Since ρ_x^* commutes with λ_t^* for every $t \in G$, so do S and U. Hence by Lemma 5.9, there exist $x_s, x_u \in G$ such that $S = \rho_{x_s}^*, U = \rho_{x_u}^*$. Thus

$$\rho_x^* = \rho_{x_s}^* \rho_{x_u}^*$$

By the same uniqueness argument as at the end of Lemma 5.9, $x = x_s x_u$. The remaining properties are clear. □

Remark 5.11. For $x \in G$,

$$\lambda_x^* = \lambda_{x_s}^* \lambda_{x_u}^*$$

is the Jordan decomposition of λ_x^* (because x_s, x_u commute with each other, the anti-homomorphism property of λ^* becomes just a homomorphism property).

Definition 5.12. For a subset S of a group G, the **centralizer** of S in G is

$$Z_G(S) = \{x \in G : xs = sx, \forall s \in S\}$$

Remark 5.13. For $g \in G$, ρ_g^* commutes with ρ_x^* if and only if it commutes with $\rho_{x_s}^*$ and $\rho_{x_u}^*$, since locally $\rho_{x_s}^*, \rho_{x_u}^*$ are polynomials in ρ_x^*. Hence

$$Z_G(x) = Z_G(x_s) \cap Z_G(x_u)$$

5.3 Jordan decompositions and morphisms

Definition 5.14. A **representation** of a group G is a group homomorphism $G \to \mathrm{GL}(V)$ for some K-vector space V. Language around representations can be confusing - sometimes the homomorphism $G \to \mathrm{GL}(V)$ is called the representation, and sometimes V is called the representation.

Remark 5.15. Let (G, A) be an algebraic group. We have previously discussed the group homomorphism

$$G \to \mathrm{GL}(A) \qquad x \mapsto \rho_x^*$$

Using the previous definition, we can say that this is a representation of G. Given $n \geq 1$, we also have the representation

$$G \to \mathrm{GL}(A^n) \cong \mathrm{GL}(A) \times \cdots \times \mathrm{GL}(A) \qquad x \mapsto (\rho_x^*, \ldots, \rho_x^*)$$

We call this a **representation of G by right translations**.

Definition 5.16. If $\rho : G \to \mathrm{GL}(V)$ is a representation, then the dual space V^* is also a representation of G, called the **dual representation**, via

$$\rho^* : G \to \mathrm{GL}(V^*) \qquad g \mapsto \rho^*(g) = \rho(g^{-1})^T$$

where the superscript T denotes transpose. More concretely, we can describe it by

$$\rho^*(g) = \rho(g^{-1})^T : V^* \to V^* \qquad \rho(g^{-1})^T(v^*) = v^* \circ \rho(g^{-1})$$

Lemma 5.17. *Let (G, A) be an algebraic group. Every representation of G is isomorphic to a representation of G on a subspace of A^n via right translations. That is, if $\rho : G \to \mathrm{GL}(V)$ is a representation, then there is an embedding $V \hookrightarrow A^n$ which is a morphism of representations of G. Being a morphism of representations means that the following square commutes for every $x \in G$.*

$$\begin{array}{ccc} V & \hookrightarrow & A^n \\ \downarrow{\scriptstyle \rho(x)} & & \downarrow{\scriptstyle (\rho_x^*, \cdots, \rho_x^*)} \\ V & \hookrightarrow & A^n \end{array}$$

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Let $v_1, \dots, v_n$ be a basis of V, and let $v_1^*, \dots, v_n^*$ be the dual basis of V^*. So for any $v \in V$, the element $v^* \in V^*$ is defined by

$$V \to V^* \qquad v = \sum_i \alpha_i v_i \mapsto v^* = \sum_i \alpha_i v_i^*$$

Recall that the matrix coefficients m_{w,v^*} for $w \in V, v^* \in V^*$, given by

$$m_{w,v^*} : G \to K \qquad m_{w,v^*}(x) = v^*(\rho(x)(w))$$

generate A as a K-algebra. Define

$$\phi : V \to A^n \qquad v \mapsto \left(m_{v,v_1^*}, \dots, m_{v,v_n^*}\right)$$

We will show that ϕ is the required embedding of G-representations. It is clear that ϕ is K-linear, since m_{w,v^*} is linear in v. First, we show that ϕ is injective. If $\phi(v) = 0$, then

$$m_{v,v_i^*} = 0 \implies m_{v,v_i^*}(x) = v_i^*(\rho(x)(v)) = 0$$

for all $i = 1, \dots, n$ and all $x \in G$. Choosing x to be the identity in G, this gives $v_i^*(v) = 0$ for all i. Since $v_1, \dots, v_n$ is a basis of V, $v = 0$. Thus $\ker \phi$ is trivial, so ϕ is injective.

Now we verify that ϕ is a morphism of representations, which means we need to show that $\phi(g \cdot v) = g \cdot \phi(v)$ for all $g \in G$. Keep in mind that $\phi(g \cdot v) = \phi(\rho(g)v)$, and $g \cdot \phi(v) = \rho_g^* \phi(v)$.

$$\phi(g \cdot v) = \phi(\rho(g)v) = \left(m_{\rho(g)v, v_1^*, \dots, m_{\rho(g)v} v_n^*}\right)$$
$$g \cdot \phi(v) = g \cdot \left(m_{v,v_1^*}, \dots, m_{v,v_n^*}\right) = \left(\rho_g^* m_{v,v_1^*}, \dots, \rho_g^* m_{v,v_n^*}\right)$$

So it comes down to verifying that $m_{\rho(g)v,v_i^*} = \rho_g^* m_{v,v_i^*}$ for all $i = 1, \dots, n$ and all $g \in G$ and all $v \in V$. By linearity in v, it suffices to show this when $v = v_j$ is a basis vector. To show $m_{\rho(g)v,v_i^*} = \rho_g^* m_{v_j,v_i^*}$ as functions, we compare their evaluations at an arbitrary $x \in G$.

$$\begin{aligned} m_{\rho(g)v,v_i^*}(x) &= v_i^*(\rho(x)\rho(g)v) = v_i^*(\rho(xg)v) \\ \rho_g^* m_{v,v_i^*}(x) &= m_{v,v_i^*}(xg) = v_i^*(\rho(xg)v) \end{aligned}$$

Thus $m_{\rho(g)v,v_i^*} = \rho_g^* m_{v,v_i^*}$, we get $\phi(g \cdot v) = g \cdot \phi(v)$, hence ϕ is an embedding of G-representations. □

Lemma 5.18. *Let G be an algebraic group and let $\rho : G \to \mathrm{GL}(V)$ be a morphism of algebraic groups. Let $x \in G$, and $x = x_s x_u$ be the Jordan decomposition. Then*

$$\rho(x) = \rho(x_s)\rho(x_u)$$

is the Jordan decomposition of $\rho(x)$ in $\mathrm{GL}(V)$.

Proof. By Lemma 5.17, we have an isomorphism of representations between V and a G-invariant subspace $V' \subset A^n$ depicted as the following commutative diagram, where $x \in G$ is arbitrary.

$$\begin{array}{ccccc} V & \xrightarrow{\cong} & V' & \hookrightarrow & A^n \\ \downarrow{\scriptstyle \rho(x)} & & \downarrow{\scriptstyle \rho_x^*|_{V'}} & & \downarrow{\scriptstyle \rho_x^*} \\ V & \xrightarrow{\cong} & V' & \hookrightarrow & A^n \end{array}$$

Let $x \in G$ and $x = x_s x_u$ be the Jordan decomposition in G, so we have $\rho(x) = \rho(x_s)\rho(x_u)$, which we want to show is the Jordan decomposition. We know that $\rho_x^* = \rho_{x_s}^* \rho_{x_u}^*$ is the Jordan decomposition of ρ_x^* in $\mathrm{GL}(A^n)$ (basically by the construction of x_s, x_u in Proposition 5.10).

That is, in the following diagram we know that the left and right sides are Jordan decompositions, but we want to verify that the central vertical composition is a Jordan decomposition.

$$\begin{array}{ccccc} V & \xrightarrow{\cong} & V' & \hookrightarrow & A^n \\ \downarrow{\scriptstyle \rho(x_u)} & & \downarrow{\scriptstyle \rho_{x_u}^*|_{V'}} & & \downarrow{\scriptstyle \rho_{x_u}^*} \\ V & \xrightarrow{\cong} & V' & \hookrightarrow & A^n \\ \downarrow{\scriptstyle \rho(x_s)} & & \downarrow{\scriptstyle \rho_{x_s}^*|_{V'}} & & \downarrow{\scriptstyle \rho_{x_s}^*} \\ V & \xrightarrow{\cong} & V' & \hookrightarrow & A^n \end{array}$$

As V' is a G-invariant subspace, $\rho_{x_s}^*|_{V'}$ and $\rho_{x_u}^*|_{V'}$ are respectively semisimple and unipotent, so

$$\rho_x^*|_{V'} = \rho_{x_s}^*|_{V'} \rho_{x_u}^*|_{V'}$$

is the Jordan decomposition in V', hence the isomorphism $V \cong V'$ forces $\rho(x) = \rho(x_s)\rho(x_u)$ to be the Jordan decomposition in $\mathrm{GL}(V)$. □

Proposition 5.19. *If G is a closed subgroup of $\mathrm{GL}(V)$, and $j : G \hookrightarrow \mathrm{GL}(V)$ is the inclusion, and $x \in G$, then*

$$j(x) = j(x_s)j(x_u)$$

is the Jordan decomposition of $j(x)$ in $\mathrm{GL}(V)$.

Proof. Immediate from previous lemma. □

Proposition 5.20. *The Jordan decomposition is preserved by a morphism of algebraic groups. That is, if $\rho : G \to G'$ is a morphism of algebraic groups, and $x = x_s x_u \in G$ is a Jordan decomposition, then*

$$\rho(x) = \rho(x_s)\rho(x_u)$$

is the Jordan decomposition of $\alpha(x)$ in G'.

Proof. This is also nearly immediate from previous results. Let $j : G' \to \mathrm{GL}(V)$ be an embedding, and consider the composition

$$G \xrightarrow{\rho} G' \xrightarrow{j} \mathrm{GL}(V)$$

For $x = x_s x_u \in G$, by Lemma 5.18,

$$j\rho(x) = j\rho(x_s)j\rho(x_u)$$

is the Jordan decomposition of $\rho j(x)$ in $\mathrm{GL}(V)$. Then by Proposition 5.19, because j is an inclusion, this implies that

$$\rho(x) = \rho(x_s)\rho(x_u)$$

is the Jordan decomposition in G'. □

5.4 Kolchin's theorem

Definition 5.21. Let G be an algebraic group. Set

$$G_S = \{x \in G : x \text{ is semisimple}\}$$
$$G_U = \{x \in G : x \text{ is unipotent}\}$$

Note that these are not necessarily subgroups or even subvarieties. There are some difficult theorems which tell us whether these are groups under additional hypotheses on G. If $G = G_U$, then G is called **unipotent**, and if $G = G_S$, then G is called **semisimple**.

Remark 5.22. If G is an abelian algebraic group, then G_S, G_U are both closed subgroups, and $G = G_S G_U$. Even better, since $G_S \cap G_U = \{e\}$, this is a direct product: $G \cong G_S \times G_U$. This is true because when G is abelian, all elements of G can be simultaneously diagonalized (after embedding G into some $\mathrm{GL}(V)$). The question then arises, under what weaker hypotheses (weaker than abelian) can this be done? One partial answer to this is given by Kolchin's theorem (Theorem 5.29).

Definition 5.23. A **semigroup** is a set with a binary operation which is associative. As a visual aid in keeping the terms groupoid, semigroup, monoid, and group straight, we include the following table. Each object has all the properties of the previous object, in addition to the new property.

Name	Property
Magma/groupoid	Set with binary operation
Semigroup	Associativity
Monoid	Identity
Group	Inverses

Definition 5.24. Let S be a groupoid. An ***S*-module** is an abelian group M with a map $S \times M \to M, (x, m) \mapsto x \cdot m$ such that for all $x, y \in S$ and $m, n \in M$,

$$x \cdot (y \cdot m) = (xy) \cdot m$$
$$x \cdot (m + n) = x \cdot m + x \cdot n$$

This is just like a group action, except without the identity requirement, since S need not have any identity. If S is a semigroup, we use this definition for S-module. If S is a monoid with identity e, we add the requirement that $e \cdot m = m$ for all m. If S is a group, we use the monoid definition. As usual, a module structure is the same as having a morphism of groupoids/semigroups/monoids

$$\rho : S \to \mathrm{End}_{\mathbb{Z}}(M)$$

If S is a groupoid, we have no requirements on this map other than preserving the binary operation, which is to say, $\rho(xy) = \rho(x) \circ \rho(y)$. If S is a semigroup nothing changes. If S is a monoid with identity e, then we require $\rho(e) = \mathrm{Id}_M$.

Definition 5.25. Let S be a groupoid and let M be an S-module. A **submodule** is a subgroup $N \subset M$ such that N is S-invariant, which is to say, N is an S-module in its own right. A module M is **simple** if it has no nonzero proper submodules.

Lemma 5.26. *Let S be a semigroup of endomorphisms of a finite dimensional vector space V such that V is a simple S-module. Then S contains a basis of $\mathrm{End}_K(V)$.*

Proof. See Lang *Algebra* page 819 [4]. □

Lemma 5.27. *Let V be an n-dimensional K-vector space, and let S be a semigroup of endomorphisms of V such that V is a simple S-module. Suppose the set*

$$\{\mathrm{tr}\, s : s \in S\} \subset K$$

is finite with cardinality r. Then S is a finite set and $|S| \leq r^{n^2}$.

Proof. By Lemma 5.26, S contains a basis $\{s_1, \ldots, s_{n^2}\}$ of $\mathrm{End}_K(V)$. Consider the map

$$\phi : S \to K^{n^2} \qquad x \mapsto (\mathrm{tr}(xs_1), \ldots, \mathrm{tr}\,(xs_{n^2}))$$

Since $xs_i \in S$ for all i, the image of ϕ is a finite set of size $\leq r^{n^2}$. Furthermore, if $x, y \in S$ such that $\phi(x) = \phi(y)$, then $\mathrm{tr}((x-y)s_i) = 0$ for all i, hence $\mathrm{tr}((x-y)z) = 0$ for all $z \in \mathrm{End}_K(V)$. Since tr is a nondegenerate bilinar form, so this is impossible, hence $\phi(x) = \phi(y) \implies x = y$, so ϕ is injective. Thus $|S| \leq r^{n^2}$. □

Definition 5.28. Let V be a finite dimensional vector space. A **flag** in V is an sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

of increasing dimension. That is, if $d_i = \dim_K V_i$, then

$$0 = d_0 < d_1 < \cdots < d_n = \dim_K V$$

A **complete flag** is a flag such that $d_i = i$ for all i. A flag that is not complete is called a **partial flag**. Another way to say that a flag is complete is that $\dim_K(V_{i+1}/V_i) = 1$ for all $i = 0, \ldots, n-1$.

Theorem 5.29 (Kolchin)**.** *Let V be an n-dimensional K-vector space, and let $G \subset \mathrm{GL}(V)$ be a closed subgroup consisting of unipotent elements. Then there is a basis of V such that G is contained in the subgroup of upper triangular matrices. That is, each element of G fixes a complete flag*

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

Proof. First we claim that there is a flag

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{r-1} \subsetneq V_r = V$$

such that $G(V_i) \subset V_i$ for all i and V_{i+1}/V_i is a simple G-module. This is an easy induction on n, as follows. If $n = 1$ this is clear. For the inductive step, Let $V' \subsetneq V$ be a proper G-invariant set of maximal dimension. By induction, there is a flag $0 = V_0 \subset \cdots \subset V_{r-1} = V'$ with V_{i+1}/V_i a simple G-module. If V/V' is not simple, then there is a G-submodule $\overline{W} \subset V/V'$ which lifts to a G-invariant subspace $W \subset V$ which contains V', contradicting maximality of V', hence V/V' is simple. Hence

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{r-1} = V' \subset V_r = V$$

is such a flag for V. This completes the induction. Now take a flag as above, and let

$$\rho_i : G \to \mathrm{GL}(V_{i+1}/V_i)$$

be the induced representation for each i. Since the elements of G are unipotent, there is a single trace value (namely the dimension), so by Lemma 5.27, $|\rho_i(G)| = 1$. Thus ρ_i is the trivial constant map. Thus $\dim V_{i+1}/V_i = 1$ since the identity keeps every subspace invariant. Thus the flag we constructed is the required complete flag (which is fixed by any $x \in G$). $\square$

Remark 5.30. Kolchin's theorem 5.29 is true for any field, we don't need K to be algebraically closed.

Remark 5.31. Examining the proof of Kolchin's theorem 5.29 and Lemma 5.27 closely, we see that a subgroup of $\mathrm{GL}_n(K)$ with r conjugacy classes must be finite with at most r^{n^2} elements. Using the contrapositive, an infinite affine algebraic group has infinitely many conjugacy classes.

A similar proof shows that if $\mathrm{char}\, K = 0$ and S is a subgroup of $\mathrm{GL}(V)$ with non nontrivial unipotent elements with r traces, then $|S| \leq r^{n^2}$ where $n = \dim_K V$, hence any torsion subgroup of $\mathrm{GL}(V)$ with bounded exponent is finite. Along similar lines, any torsion subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is finite of order $\leq (2n+1)^{n^2}$, by a result of Burnside.

Remark 5.32. By taking any nonzero vector in V_0 from the complete flag of Kolchin's theorem 5.29, we obtain a vector which is an eigenvector for every element of G.

6 Diagonalizable groups

6.1 Unipotent, nilpotent, and solvable groups

Definition 6.1. Let G be a group. Let $G_0 = G^{(0)} = G$, and define $G_i, G^{(i)}$ recursively by

$$G_{i+1} = [G, G_i] \qquad G^{(i+1)} = [G^{(i)}, G^{(i)}]$$

Note that

$$G_i \supset G_{i+1} \qquad G^{(i)} \supset G^{(i+1)}$$

(in fact, these are normal subgroups). The group G is **nilpotent** if G_i is trivial for some i, and G is **solvable** if $G^{(i)}$ is trivial for some i. That is, G is nilpotent if

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = 1$$

terminates and G is solvable if

$$G = G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(n)} = 1$$

terminates.

Remark 6.2. Every nilpotent group is solvable. A subgroup of a solvable (respectively nilpotent) group is solvable (nilpotent). Similar statements hold for homomorphic images, etc.

Example 6.3. Let K be any field. The subgroup of $\mathrm{GL}_n(K)$ of upper triangular matrices is solvable, but not nilpotent. The subgroup of upper triangular matrices with 1's on the diagonal is nilpotent (and hence solvable as well).

Corollary 6.4. *Every unipotent algebraic group is nilpotent.*

Proof. By Kolchin's theorem, if G is unipotent, it is isomorphic to a subgroup of upper triangular matrices with 1's on the diagonal, which is a nilpotent group. □

Remark 6.5. Let G be an algebraic group acting on an affine variety V. Recall that each orbit of G is open in its closure, and minimal dimension orbits are closed. The next proposition gives a stronger version of this when G is unipotent.

Proposition 6.6. *Let G be a unipotent algebraic group acting on a variety V. Then every orbit is closed.*

Proof. We have a morphism $\alpha : G \times V \to V$ of affine varieties, inducing a K-algebra homomorphism

$$\alpha^* : K[V] \to K[G] \otimes_K K[V]$$

For each $x \in G$, define the K-algebra endomorphism

$$x^* : K[V] \to K[V] \qquad (x^* f)(v) = f(x^{-1}v)$$

Note that $(xy)^* = x^* y^*$, so we have a group homomorphism

$$G \to \mathrm{Aut}_K(K[V]) \qquad x \mapsto x^*$$

As in Lemma 2.10, we can show that the vector space

$$W_f = \langle x^* f : x \in G \rangle_K$$

is finite dimensional over K, and invariant under G. So the action of G on $K[V]$ is locally finite. As in Lemma 2.13, the morphism

$$G \to \mathrm{GL}(W_f) \qquad x \mapsto x^*|_{W_f}$$

is a morphism of algebraic groups. Now, since $x \in G$ is unipotent, $x^*|_{W_f}$ is unipotent for any $x \in G, f \in K[V]$. Also, $K[V] = \sum_f W_f$, so x^* is locally unipotent for each $x \in G$.

Let $\mathcal{O} \subset V$ be an orbit of G, and suppose that $\mathcal{O}$ is not closed. It is open in its closure $\overline{\mathcal{O}}$, and $\overline{\mathcal{O}} \setminus \mathcal{O}$ is a union of smaller dimensional orbits (Corollary 3.6). Hence $\overline{\mathcal{O}} \setminus \mathcal{O}$ is a proper closed subset of $\overline{\mathcal{O}}$, so there exists $f \in K\left[\overline{\mathcal{O}}\right]$ such that $f|_{\overline{\mathcal{O}}} \neq 0$ but $f|_{\overline{\mathcal{O}} \setminus \mathcal{O}} = 0$. Fix such an f. We will say f has property $(*)$. Since $\overline{\mathcal{O}} \setminus \mathcal{O}$ is a union of orbits, $x^* f$ (for any $x \in G$) also has property $(*)$.

$$(x^* f)|_{\overline{\mathcal{O}}} \neq 0 \qquad (x^* f)|_{\overline{\mathcal{O}} \setminus \mathcal{O}} = 0$$

Since G is unipotent,

$$\left\{ x^*|_{W_f} : x \in G \right\}$$

is a unipotent subgroup of $\mathrm{GL}(W_f)$, so by Kolchin's theorem 5.29 and Remark 5.32, there is a common (nonzero) eigenvector $f_0 \in K\left[\overline{\mathcal{O}}\right]$. Since $f_0|_{\overline{\mathcal{O}}} \neq 0$ and every element of W_f is zero on $\overline{\mathcal{O}} \setminus \mathcal{O}$, f_0 also has property $(*)$. Since G is unipotent, the only possible eigenvalue for f_0 is 1, so $x^* f_0 = f_0$ for all $x \in G$.

Because $x^* f_0 = f_0$, $f_0(x^{-1}) = f_0(v)$ for all $x \in G$, so f_0 is constant on each orbit in $\overline{\mathcal{O}}$. Hence f_0 is constant on $\mathcal{O}$, since $\mathcal{O}$) is an open orbit in $\overline{\mathcal{O}}$. Hence f_0 is constant on $\overline{\mathcal{O}}$, since $\mathcal{O}$ is dense in $\overline{\mathcal{O}}$, and $\left\{ v \in \overline{\mathcal{O}} : f_0(v) = \lambda \right\}$ is a closed subset of $\overline{\mathcal{O}}$ containing $\mathcal{O}$. Since $f_0 = 0$ on $\overline{\mathcal{O}} \setminus \mathcal{O}$, $f_0 = 0$ on $\overline{\mathcal{O}}$. This is a contradiction, so we conclude that $\mathcal{O}$ was closed to begin with. □

6.2 Diagonalizable groups and characters

Definition 6.7. An affine algebraic group G is **diagonalizable** if G is abelian, and every $x \in G$ is semisimple ($G = G_S$).

Example 6.8. The algebraic group $\mathbb{G}_m \cong K^\times$ is diagonalizable. (The coordinate ring is $K[x, x^{-1}]$.)

Definition 6.9. Let G be an algebraic group. A **character** of G is an algebraic group homomorphism $\chi : G \to \mathbb{G}_m$. The **group of characters** of G is

$$X(G) = \mathrm{Hom}(G, \mathbb{G}_m)$$

Note that $X(G)$ is an abelian group under pointwise multiplication. That is, multiplication is defined for $f, g \in X(G)$ by

$$(fg)(x) = f(x)g(x)$$

where $x \in G$.

Example 6.10. We show that the character group of $\mathbb{G}_m$ is (isomorphic to) $\mathbb{Z}$. First, note that $K[\mathbb{G}_m] \cong K[x, x^{-1}]$. For $\alpha \in X(\mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$, consider $\alpha^* : K[x, x^{-1}] \to K[x, x^{-1}]$. Since α^* is a K-algebra homomorphism, it is determined by the value of x, so let

$$\alpha^*(x) = \frac{f(x)}{x^n}$$

where $f \in K[x]$ is some polynomial and $n \in \mathbb{Z}$. We claim that f has no nonzero roots. Suppose $\lambda \in \mathbb{G}_m$ is a (nonzero) root of $f(x)$. Then for any $h \in K[\mathbb{G}_m] \cong K[x, x^{-1}]$, write h as a (finite) sum

$$h = \sum_{i \in \mathbb{Z}} \beta_i x^i \qquad \beta_i \in K$$

then

$$\alpha^*(h) = \alpha^* \sum_i \beta_i x^i = \sum_i \beta_i (\alpha^* x)^i = \sum_i \beta_i \left(\frac{f(x)}{x^n} \right)^i$$

Evalutaing $\alpha^*(h)$ at λ, each term of the sum is zero, so

$$0 = \alpha^*(h)(\lambda) = h(\alpha(\lambda))$$

Since $h \in K[\mathbb{G}_m]$ was arbitrary, this says that all elements of $K[\mathbb{G}_m]$ take the same value at $\alpha(\lambda)$, which contradicts the separation of points axiom. Hence no such nonzero λ exists. Thus $\alpha^*(x) = cx^m$ for some $m \in \mathbb{Z}, c \in K$. Since $\alpha^*(1) = 1$, we get $c = 1$, so

$$\alpha : \mathbb{G}_m \to \mathbb{G}_m \qquad g \mapsto g^m$$

Thus we obtain an isomorphism of groups

$$X(\mathbb{G}_m) \xrightarrow{\cong} \mathbb{Z} \qquad (g \mapsto g^m) \mapsto m$$

Proposition 6.11. *Let G be an algebraic group. The following are equivalent.*

1. *G is diagonalizable.*
2. *G is isomorphic to a closed subgroup of $D_n = (\mathrm{GL}_1(K))^n = (\mathbb{G}_m)^n$.*
3. *$K[G]$ is spanned (as a K-vector space) by characters of G.*

Proof. (1) $\implies$ (2) G is an affine group, so it is linear, so we have an embedding $G \hookrightarrow \mathrm{GL}(V)$ where V is a finite dimensional K-vector space. Since G is diagonalizable, it is abelian, so there is a basis of V which simultaneously diagonalizes the image of G in $\mathrm{GL}(V)$. So G is isomorphic to a closed subgroup of $(\mathbb{G}_m)^n$.

(2) $\implies$ (3) Let G be a closed subgroup of D_n. For $m_1, \ldots, m_n \in \mathbb{Z}$,

$$G \to \mathbb{G}_m \qquad x = \text{diag}(x_{11}, \ldots, x_{nn}) \mapsto x_{11}^{m_1} \cdots x_{nn}^{m_n}$$

is a character of G. $K[G]$ consists of Laurent polynomials in the x_{ii}, hence the characters span $K[G]$.

(3) $\implies$ (1) Let $f : G \to \mathbb{G}_m$ be a character, so $f(xy) = f(x)f(y)$ for $x, y \in G$. Thus

$$\rho_y^*(f)(x) = f(xy) = f(x)f(y) \implies \rho_y^* f = f(y)f$$

so each character $f \in X(G)$ is an eigenvector for ρ_y^* (where $y \in G$ is arbitrary). Since the characters span $K[G]$, ρ_y^* is semisimple, hence $y \in G$ is semisimple, so $G = G_S$. Also, $\rho_y^* \rho_z^* = \rho_z^* \rho_y^*$ (because they act by scalar multiplication on $K[G]$) so G is abelian as well, hence G is diagonalizable. □

Lemma 6.12 (Linear independence of characters, due to Artin)**.** *Distinct characters of a group G into the multiplicative group of a field $K^\times$ are linearly independent as K-valued functions on G.*

Proof. See Lang [4] or other sources. □

Remark 6.13. If G is diagonalizable, then $X(G)$ spans $K[G]$ as a K-vector space, and it is a linear indepenedent set, so $X(G)$ is a basis for $K[G]$.

Proposition 6.14. *Let G be a diagonalizable group. Then $X(G)$ is finitely generated (abelian).*

Proof. By Proposition 6.11, $K[G]$ is spanned by characters. Since $K[G]$ is a finitely generated K-algebra, there exist $\chi_1, \ldots, \chi_n \in X(G)$ which generate $K[G]$ as a K-algebra. Let $H \subset \chi(G)$ be the subgroup generated by $\chi_1, \ldots, \chi_n$.

$$H = \{\chi_1^{r_1} \ldots \chi_n^{r_n} : r_i \in \mathbb{Z}\}$$

Now any element of $K[G]$ is a finite linear combination of elements of H. We claim that $H = X(G)$, which will show that $\chi(G)$ is finitely generated. Let $\chi \in X(G) \subset K[G]$, and suppose $\chi \notin H$. Since H generates $K[G]$ as a K-algebra, we may write χ as a linear combination

$$\chi = \sum_j a_j \eta_j$$

where $\eta_j \in H, a_j \in K$, and we combine terms so that all of the η_j are distinct. We rewrite this as

$$0 = \chi - \sum_j a_j \eta_j$$

Since $\chi \notin H$, χ is distinct from all of the η_j. But by linear independence of characters, all of the a_j must be zero so $\chi = 0$, but this is not a character, so we have a contradiction. Thus $\chi \in H$, and $H = K[G]$, so $K[G]$ is finitely generated. □

Proposition 6.15. *Let G be a diagonalizable group and H a closed subgroup of G. Let $\chi \in X(H)$. Then χ extends to a character $\widetilde{\chi} \in X(G)$, and*

$$H = \bigcap_{\chi \in X(G)} \ker \chi$$

(this intersection statement is not right)

Proof. It suffices to prove this in the case $G \cong \mathbb{G}_m^n$, since then one obtains an extension to G by considering $H \subset G \subset \mathbb{G}_m^n$ and restricting the extension to $\mathbb{G}_m^n$ to G.

We start by addressing the case $n = 1$, the general case will be very similar. Let $\iota : H \hookrightarrow \mathbb{G}_m$ be the inclusion, and let $\chi : H \to \mathbb{G}_m$ be a character of H. Finding an extension $\beta : \mathbb{G}_m \to \mathbb{G}_m$ of χ is equivalent to finding an extension β^* in the corresponding K-algebra homomorphism diagram,

$$\begin{array}{ccc} H & \xhookrightarrow{\iota} & \mathbb{G}_m \\ {\scriptstyle \chi}\downarrow & \swarrow & \\ \mathbb{G}_m & & \end{array} \qquad \begin{array}{ccc} K[H] & \xleftarrow{\iota^*} & K[\mathbb{G}_m] \\ {\scriptstyle \chi^*}\uparrow & \nearrow {\scriptstyle \beta^*} & \\ \mathbb{G}_m & & \end{array}$$

Now recall that $K[\mathbb{G}_m] = K[x, x^{-1}]$, and $K[H] = K[x, x^{-1}]/I$ for some ideal $I \subset K[x, x^{-1}]$. Also, ι^* is just the quotient map, or alternatively, the restriction map, since for $f \in K[\mathbb{G}_m] = K[x, x^{-1}]$ and $h \in H$,

$$(\iota^* f)(h) = f \circ \iota(h) = f(h) \implies \iota^* f = f|_H = \operatorname{res} f$$

So we redraw our K-algebra diagram as

$$\begin{array}{ccc} K[x, x^{-1}]/I & \xleftarrow{\text{res}} & K[x, x^{-1}] \\ {\scriptstyle \chi^*}\uparrow & \nearrow {\scriptstyle \beta^*} & \\ K[x, x^{-1}] & & \end{array}$$

The notation is a bit funky, but $x \in K[x, x^{-1}]$ is just the inclusion map $x : \mathbb{G}_m = K^\times \hookrightarrow K$. So for $h \in H$,

$$(\chi^* x)(h) = x \circ \chi(h) = \chi(h) \implies \chi^* x = \chi$$

As $K[x, x^{-1}]$ is generated as a K-algebra by x and x^{-1}, to define β^* is suffices to define it on x and verify that $\beta^* x$ is invertible, so that we can set $\beta^*(x^{-1}) = (\beta^* x)^{-1}$. We can write $\chi \in K[x, x^{-1}]/I$ as

$$\chi = \widetilde{\chi} + I \qquad \widetilde{\chi} \in K[x, x^{-1}]$$

That is, we choose a lift $\widetilde{\chi}$ of χ. As χ is nowhere vanishing (it maps to $\mathbb{G}_m$), $\chi \neq 0$ so $\widetilde{\chi}$ is not in I, and in particular, $\widetilde{\chi} : K[x, x^{-1}]$ is nowhere vanishing, so it is invertible [3]. So we define $\widetilde{\beta}^*(x) = \widetilde{\chi}$ and we have defined a K-algebra homomorphism making this diagram commute.

[3] Here invertible does not mean that it has an inverse function. It just means it is an invertible element of $K[x, x^{-1}]$.

This K-algebra homomorphism then corresponds to a morphism of varieties $\beta : \mathbb{G}_m \to \mathbb{G}_m$ extending χ.

To prove this in the more general case of $G = \mathbb{G}_m^n$, use roughly the same argument. It is sufficient to find a K-algebra homomorphism $K[\mathbb{G}_m^n] \to K[\mathbb{G}_m^n]$ extending $\chi^* : K[\mathbb{G}_m^n] \to H$. Now $K[\mathbb{G}_m^n] = K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ and $K[H]$ is once again a quotient of this. In this setting,

$$(\chi^* x_i)(h) = x_i \circ \chi(h)$$

By the same process as in the $n = 1$ case, define $\beta^*(x_i)$ to be a lift of $x_i \circ \chi(h)$, and things work out as in the $n = 1$ case. □

Remark 6.16. If G_1, G_2 are diagonalizable, then $G_1 \times G_2$ is diaganalizable, and

$$X(G_1 \times G_2) \cong X(G_1) \times X(G_2)$$

Definition 6.17. For a field K, let

$$p(K) = \begin{cases} \operatorname{char} K & \text{if } \operatorname{char} K \neq 0 \\ 1 & \text{if } \operatorname{char} K = 0 \end{cases}$$

Proposition 6.18. *If G is a diagonalizable algebraic group over a field with characteristic $p(K) \geq 2$, then $X(G)$ has no p-torsion.*

Proof. Suppose $\chi \in X(G)$ such that $\chi^p = 1$. Then for all $x \in G$,

$$\chi(x)^p = \chi^p(x) = 1$$

the above being an equality in K. Since K has characteristic p, $x^p - 1 = (x-1)^p$, so the only pth root of unity is 1, hence $\chi(x) = 1$, so $\chi = 1$. □

Proposition 6.19. *Let K be a field with $p = p(K)$, and let X be a finitely generated abelian group with torsion prime to p. Then there exists a diagonalizable group G (over K) with $X(G) \cong X$.*

Proof. Since $X(G_1 \times G_2) \cong X(G_1) \times X(G_2)$, using the classification of finitely generated abelian groups, it suffices to prove this when X is cyclic. If X is infinite cyclic, then we have already shown in Example 6.10 that $X(\mathbb{G}_m) \cong \mathbb{Z} \cong X$, so it just remains to address the case where $X \cong \mathbb{Z}/n\mathbb{Z}$ is finite cyclic and $\gcd(n, p) = 1$.

Let $G \subset \mathbb{G}_m \cong K^\times$ be the group of nth roots of unity. Then G is a closed subgroup of $\mathbb{G}_m$ and hence diagonalizable. Let $\alpha \in X(G)$, so $\alpha : G \to \mathbb{G}_m$ is a morphism of algebraic groups. The image of α must be in the n-torsion of $\mathbb{G}_m$, which is G, so we may view α as a morphism $G \to G$. Since G is cyclic, $\alpha : G \to G$ must be a map of the form $\zeta \mapsto \zeta^n$ for some $m \in \mathbb{Z}$, hence $X(G) \cong \mathbb{Z}/n\mathbb{Z}$. □

Remark 6.20. Let G, G' be diagonalizable groups, and let $\phi^* : X(G') \to X(G)$ be a group homomorphism. Since $X(G')$ is a K-basis for $K[G']$, ϕ^* extends by linearity to a K-linear map $\phi^* : K[G'] \to K[G]$. Even better, we can show that this extension of ϕ^* is a K-algebra homomorphism, as follows. Let $f, g \in K[G']$, and write them (uniquely) as

$$f = \sum_i a_i \lambda_i \qquad g = \sum_j b_j \mu_j$$

with $a_i, b_j \in K, \lambda_i, \mu_j \in X(G')$. Then using the fact that ϕ^* is a group homomorphism,

$$\begin{aligned}
\phi^*(fg) &= \phi^* \left(\left(\sum a_i \lambda_i \right) \left(\sum b_j \mu_j \right) \right) \\
&= \phi^* \left(\sum a_i b_j \lambda_i \mu_j \right) \\
&= \sum a_i b_j \phi^*(\lambda_i \mu_j) \\
&= \sum a_i b_j \phi^*(\lambda_i) \phi^*(\mu_j) \\
&= \left(\sum_i a_i \phi^*(\lambda_i) \right) \left(\sum_j b_j \phi^*(\mu_j) \right) \\
&= \phi^*(f) \phi^*(g)
\end{aligned}$$

Hence ϕ^* extends uniquely to a K-algebra homomorphism.

Proposition 6.21. *Let G, G' be diagonalizable groups. Given any homomorphism $\phi^* : X(G') \to X(G)$, there exists a unique morphism of algebraic groups $\alpha : G \to G'$ such that the K-linear extension of $\phi^* : K[G'] \to K[G]$ is equal to $\alpha^* : K[G'] \to K[G]$.*

Proof. Extend ϕ^* to a morphism of K-algebras $\phi^* : K[G'] \to K[G]$ as in the previous remark. For each $x \in G$, the composition $\mathrm{ev}_x \circ \phi^*$ is a K-algebra homomorphism $K[G] \to K$, so it is the evaluation map at some (unique) point, which we denote by $\alpha(x)$. Thus we obtain a map $\alpha : G \to G'$ satisfying $\mathrm{ev}_{\alpha(x)} = \mathrm{ev}_x \circ \phi^*$ for all $x \in G$. Then it is clear that $\alpha^* = \phi^*$, so α is a morphism of varieties.

It remains to show that α is a group homomorphism. Consider $f \in X(G')$, and $x, y \in G$. Then

$$f(\alpha(xy)) = (\alpha^* f)(xy) = (\alpha^* f)(x) \cdot (\alpha^* f)(y) = \Big(f \circ \alpha(x) \Big) \cdot \Big(f \circ \alpha(y) \Big) = f(\alpha(x)\alpha(y))$$

Since $f \in X(G')$ was arbitrary and $X(G')$ is a basis of $K[G']$, the above equality holds for all $f \in K[G']$. Then by the separation of points axiom, $\alpha(xy) = \alpha(x)\alpha(y)$ for all x, y, hence α is a morphism of algebraic groups.

Finally, we need to show that α is unique. Suppose $\alpha, \beta : G \to G'$ are such that $\alpha^* = \beta^*$. Then for all $f \in K[G']$ and $x \in G$,

$$f(\alpha(x)) = \alpha^* f(x) = \beta^* f(x) = f(\beta(x))$$

Again using the separation of points axiom, this implies $\alpha(x) = \beta(x)$, so $\alpha = \beta$. □

At this time, we take a moment to remind you that everything is happening over a fixed algebraically closed field K.

Remark 6.22. Given a digaonalizable group G over K, we have shown that $X(G) = \text{Hom}(G, \mathbb{G}_m)$ is a finitely generated abelian group with torsion prime to $p(K)$. Given a morphism $\alpha : G \to G'$ of diagonalizable algebraic groups, there is the usual induced map associated with the (contravariant) hom functor,

$$X(G') \to X(G) \qquad f \mapsto f \circ \alpha = \alpha^*(f)$$

Thus the assignment $G \mapsto X(G)$ is a (contravariant) functor. Proposition 6.19 shows that this functor is essentially surjective ("surjective" in terms of isomorphism classes of objects). Proposition 6.21 shows that this functor is fully faithful (induces isomorphisms on hom sets). Thus it is an equivalence of categories. By Remark 6.16, it also preserves binary products. We state this as a theorem for reference.

Theorem 6.23 (Equivalence of categories)**.** *The assignment*

$$\{\text{diagonalizable groups}\} \to \{\text{finitely generated abelian groups with torsion prime to } p(K)\}$$
$$G \mapsto X(G) = \text{Hom}(G, \mathbb{G}_m)$$

is an equivalence of categories.

Remark 6.24. Because $X(-)$ is an equivalence of categories, it is an exact functor.

6.3 Tori

Proposition 6.25. *Let G be a diagonalizable group. The following are equivalent.*

1. *G is connected.*
2. *$G \cong (\mathbb{G}_m)^n$ for some n.*
3. *$X(G)$ is free abelian.*

Proof. We prove (1) $\implies$ (3) $\implies$ (2) $\implies$ (1).
(1) $\implies$ (3) As G is connected, $K[G]$ is an integral domain (Lemma 2.21). So $X(G) \subset K[G]$ cannot have any torsion, so $X(G)$ is free abelian.
(3) $\implies$ (2) By the previous theorem, if $X(G)$ is free abelian, then $G \cong (\mathbb{G}_m)^n$.
(2) $\implies$ (1) Clear.

□

Definition 6.26. A diagonalizable group satisfying the above equivalent conditions is called a **torus**.

Remark 6.27. If G is a diagonalizable group, then G^0 is connected so it is a torus, hence $G^0 \cong (\mathbb{G}_m)^n$ for some n. Recall that G^0 is a normal subgroup of finite index (Proposition 2.25), and consider the short exact sequence

$$1 \to G^0 \to G \to G/G^0 \to 1$$

Since G^0 has finite index, G/G^0 is finite and discrete. Now apply the (contravariant) exact equivalence of categories $X(-)$.

$$1 \to X(G/G^0) \to X(G) \to X(G^0) \to 1$$

Since $X(G^0)$ is free abelian (hence projective), this sequence splits, so the original sequence is also split. Hence we can decompose any diagonalizable group G as a product of the torus identity component G^0 with a finite abelian group with torsion prime to $p(K)$.

$$G \cong G^0 \times G/G^0$$

Proposition 6.28. *Let G be a diagonalizable algebraic group.*

1. *The elements of finite order are dense in G.*
2. *For an integer n, there exist only finitely many elements of order n.*
3. *If K is not finite, and G is a torus over K, then there is an element $x \in G$ whose powers are dense in G.*

Proof. (1) Decompose G as

$$G \cong (\mathbb{G}_m)^n \times \prod_{i=1}^{m} \mathbb{Z}/n_i\mathbb{Z}\langle \alpha_i \rangle$$

Letting $x_1, \ldots, x_n, y_1, \ldots, y_m$ be free variables,

$$K[G] \cong K[x_1, \ldots, x_n, y_1, \ldots, y_m]/I \cong K[x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m]$$

where I is the ideal generated by $\alpha_1^{n_1} - 1, \ldots, \alpha_m^{n_m} - 1$. We claim that there is no $f \in K[G]$ that vanishes on all $(n+m)$-tuples of roots of unity, but not on all of G. If this is true, then there is no closed subset of G containing all roots of unity which does not contain all of G, which is to say, the elements of finite order are dense.

We begin by addressing the case $n = 1$. Let $f \in K[G]$ and suppose f vanishes at all tuples of roots of unity, and write it as

$$f(x, \alpha_1, \ldots, \alpha_m) = \sum_i a_i \alpha(i) x^i$$

where $a_i \in K$ and $\alpha(i)$ is some polynomial in the variables α_i. Choose various roots of unity $\zeta_1, \ldots, \zeta_m \in K^\times$ and evaluate $f(x, \zeta_1, \ldots, \zeta_m)$, we obtain a single variable polynomial $f(x) \in K[x]$, which vanishes at all roots of unity in K. Since K is algebraically closed, there are infinitely many of these, hence f vanishes on all of G.

If $n > 1$, we can do a similar trick to reduce to the $n-1$ case by writing f as a polynomial in one fewer variable, so an induction completes the proof.

(2) As $G \subset \mathbb{G}_m^n$ for some n, it suffices to show that $\mathbb{G}_m^n$ has finitely many elements of a given finite order. Since n is finite, it suffices to show that $\mathbb{G}_m$ has finitely many elements of a given finite order. This is obvious, because $\mathbb{G}_m \cong K^\times$ and $K^\times$ has at most m mth roots of unity.

(3) Proof not done in class. ∎

Remark 6.29. Let T be a torus, and consider elements $\alpha_1, \dots, \alpha_r \in X(T)$. To say that these elements are "linearly inpendent" is mildly ambiguous, since they may be linearly independent as elements of the free abelian group $X(T)$, or they may be linearly independent as functions $T \to K$. However, one of these is strictly weaker, since being linearly independent as elements of $X(T)$ implies they are also linearly indepent as functions on T.

Proposition 6.30. *Let T be a torus, and let $\alpha_1, \dots, \alpha_r \in X(T)$ be linearly independent as elements of $X(T)$. Let $c_1, \dots, c_r \in \mathbb{G}_m$. Then there exists $t \in T$ such that $\alpha_i(t) = c_i$ for all i.*

Proof. Consider the function

$$f : T \to \mathbb{G}_m^r \qquad x \mapsto (\alpha_1(x), \dots, \alpha_r(x))$$

This is a morphism of algebraic groups, and the proposition is equivalent to surjectivity of f. Since $\alpha_1, \dots, \alpha_r$ are linearly independent, they generate a free abelian subgroup of $X(T)$, so al of the monomials in them are distinct and linearly independent as functions on T. Hence

$$f^* : K[\mathbb{G}_m^r] \to K[T]$$

is injective. Thus f as dense image in $\mathbb{G}_m^r$. Since f is a morphism of algebraic groups, $f(T)$ is closed in $\mathbb{G}_m^r$. Thus f is surjective. □

6.4 Rigidity theorem, stabilizers, normalizers, and centralizers

Theorem 6.31 (Rigidity Theorem)**.** *Let V be a connected affine variety and H, H' be algebraic groups satisfying*

1. *Elements of finite order in H are dense (in H).*
2. *Elements of any given finite order in H' are finite.*

Let $\alpha : V \times H \to H'$ be a morphism of varieties, such that for each $v \in V$,

$$\alpha_v : H \to H' \qquad h \mapsto \alpha(v, h)$$

is a homomorphism of algebraic groups. Then

$$V \to \mathrm{Hom}(H, H') \qquad v \mapsto \alpha_v$$

is a constant map. That is, for $v, v' \in V$, $\alpha_v = \alpha_{v'}$.

As a slogan, one may remember the previous theorem as saying that under certain circumstances (V is connected, H, H' have properties 1,2), that *there is no nonconstant family of algebraic group homomorphisms parametrized by a variety V.* Stated this way, the theorem is quite amazing and profound, so it is a surprise that the proof is relatively simple.

Proof. For $h \in H$, consider the morphism of varieties

$$\alpha_h : V \to H' \qquad v \mapsto \alpha_v(h)$$

and let

$$S_h = \operatorname{im} \alpha_h = \{\alpha_v(h) : v \in V\}$$

Suppose $h \in H$ has finite order. Then for each $v \in V$, $|\alpha_v(h)|$ divides $|h|$, so the order of any element in S_h is bounded above by $|h|$. Hence S_h is finite (by our finiteness hypothesis on H').

Since V is connected, S_h is connected. Since S_h is a connected finite set, it is a point. Hence $\alpha_v(h) = \alpha_{v'}(h)$ for all $v, v' \in V$, provided h has finite order. Since such elements are dense in H by hypothesis, we are nearly done.[4] If we show that given $v, v' \in V$, the set

$$U_{v,v'} = \{h \in H : \alpha_v(h) = \alpha_{v'}(h)\}$$

is closed in H, then since it also contains a dense subset of H (the finite order elements), it must be all of H. So it just remains to show $U_{v,v'} \subset H$ is closed. To see this, consider the morphism of varieties

$$\beta : H \to H' \times H' \qquad h \mapsto (\alpha_v(h), \alpha_{v'}(h))$$

Then $U_{v,v'}$ is the preimage of the diagonal $\Delta(H') = \{(h', h') : h' \in H'\}$, which is a closed subset, so $U_{v,v'}$ is closed.

$$U_{v,v'} = \beta^{-1}(\Delta(H'))$$

□

Remark 6.32. By Proposition 6.28, the hypotheses of the Rigidity theorem 6.31 are satisfied whenever H, H' are diagonalizable.

Definition 6.33. Let G be a group and $H \subset G$ any subset. The **normalizer** and **centralizer** of H in G are respectively

$$N_G(H) = \left\{g \in G : gHg^{-1} = H\right\}$$
$$Z_G(H) = \left\{g \in G : ghg^{-1} = h, \ \forall h \in H\right\}$$

Note that $Z_G(H) \subset N_G(H)$.

Lemma 6.34. *Let G be an algebraic group and $H \subset G$ a (closed, algebraic) subgroup. Then $N_G(H)$ and $Z_G(H)$ are (closed, algebraic) subgroups of G.*

[4] When I first saw this proof, I thought we were done at this point, since I recalled a result which says that a continuous function is determined by its values on a dense subset. However, I had forgotten that this result needs the additional hypothesis that the space involved in Hausdorff, which is not true for Zariski topologies.

Proof. For $h \in H$, consider

$$G \to G \qquad g \mapsto [g, h] = ghg^{-1}h^{-1}$$

This is a morphism of algebraic groups, with kernel $Z_G(h)$. Thus $Z_G(h)$ is a closed subgroup of G. Then observe

$$Z_G(H) = \bigcap_{h \in H} Z_G(h)$$

hence $Z_G(H)$ is a closed subgroup of G. For normalizers, see a proof in Humphreys [3], page 59. □

Corollary 6.35. *Let G be an algebraic group and $H \subset G$ a (closed) diagonalizable subgroup. Then*

1. $N_G(H)^0 = Z_G(H)^0$. *In particular, if G is connected and H is normal, then H is central.*
2. $N_G(H)/Z_G(H)$ *is finite.*

Proof. (1) Apply the Rigidity theorem 6.31 in the case $V = N_G(H)^0, H = H, H' = H$ with

$$\alpha : N_G(H)^0 \times H \to H \qquad (v, h) \mapsto vhv^{-1}$$

By the Rigidity theorem, α_v does not depend on v, so $vhv^{-1} = ehe^{-1} = h$ for all $v \in N_G(H)^0$. Hence $v \in Z_G(H)$, so $N_G(H)^0 \subset Z_G(H)^0$. The reverse inclusion is clear.

(2) Using(1), $N_G(H)/Z_G(H)$ is the homomorphic image of the finite group $N_G(H)/N_G(H)^0$, as depicted below.

$$N_G(H) \to N_G(H)/N_G(H)^0 \cong N_G(H)/Z_G(H)^0 \to N_G(H)/Z_G(H)$$

The first and last arrows are both quotient maps, and the middle two terms are finite groups, so the last term is also finite. □

Definition 6.36. Let G be a group acting on a set V. Given a set $H \subset G$, the **fixed points** of H are

$$\text{fix}(H) = \{v \in V : hv = v, \ \forall h \in H\}$$

Given a subset $U \subset V$, the **pointwise stabilizer** of U is

$$\text{stab}_\text{p}(U) = \{g \in G : gu = u, \ \forall u \in U\}$$

Note that stab(U) is a subgroup of G. The **setwise stabilizer** of U is

$$\text{stab}_\text{s}(U) = \{g \in G : gU = U\}$$

Disucssions of stabilizers in other sources are rarely this careful to distinguish, and the reader often has to figure out which type of stabilizer is meant by the author. It is more common to use "stabilizer" to refer to setwise stabilizer. However, in what follows, we will be more concerned with pointwise stabilizers.

Proposition 6.37. *Let G be a diagonalizable algebraic group acting on an affine variety V.*

1. *Finitely many subsets of V occur as fixed point sets of subsets of G.*

2. *Finitely many subgroups of G occur as pointwise stabilizers of subsets of V.*

3. *If G is connected, then there exists $x \in G$ such that $xv = v$ implies $yv = v$ for all $y \in G$. The set of all such x is dense.*

Proof. Before addressing any of (1),(2),(3) directly, we need a lot of set up. Since G is diagonalizable, it acts on $K[V]$ by semisimple K-algebra automorphisms,

$$G \times K[V] \to K[V] \qquad (x, f) \mapsto x \cdot f = \rho_x^* f$$

Since G is abelian, its image in $\operatorname{Aut}(K[V])$ is abelian, so there is a finite set $\{f_1, \ldots, f_n\} \subset K[V]$ consisting of simultaneous eigenvectors for G, and so that $\{f_1, \ldots, f_n\}$ generates $K[V]$ as a K-algebra. For each i, and each $x \in G$, let $\chi_i(x)$ be the eigenvalue associated to f_i.

$$x \cdot f_i = \chi_i(x) f_i$$

That is, for $v \in V$,

$$(x \cdot f_i)(v) = (\rho_x^* f_i)(v) = f_i(xv) = \chi_i(x) f_i(v)$$

So for each i we have a function

$$\chi_i : G \to K \qquad x \mapsto \chi_i(x)$$

Because x is semisimple, $\chi_i(x) \neq 0$, so the image of χ_i lands in $\mathbb{G}_m$. Also, for $x, y \in G$ and $v \in V$,

$$\chi_i(xy) f_i(v) = f_i(xyv) = f_i(x \cdot (yv)) = (x \cdot f_i)(yv) = \chi_i(x) f_i(yv) = \chi_i(x)\chi_i(y) f_i(v)$$

thus $\chi_i(xy) = \chi_i(x)\chi_i(y)$, so χ_i is a character of G, $\chi_i \in X(G)$. Now, for any $x \in G$ and $v \in V$,

$$xv = v \iff f_i(xv) = f_i(v), \ \forall i$$

because the f_i generate $K[V]$ as a K-algebra. Rearranging the equation on the right side,

$$\begin{aligned} xv = v &\iff f_i(xv) = f_i(v), \ \forall i \\ &\iff \chi_i(x) f_i(v) = f_i(v) \ \forall i \\ &\iff (\chi_i(x) - 1) f_i(v) = 0 \ \forall i \end{aligned}$$

Finally we have done enough setup and we can prove (1). Let $S \subset G$ be any subset, and let

$$\begin{aligned} W_S &= \operatorname{fix}(S) = \{v \in V : xv = v, \ \forall x \in S\} \\ J_S &= \{i : 1 \le i \le n, \chi_i|_S \neq 1\} \\ \widetilde{W}_S &= \{v \in V : f_i(v) = 0, \ \forall i \in J\} \end{aligned}$$

We claim that $W_S = \widetilde{W}_S$. If $v \in W_S$, then $(\chi_i(x) - 1)f_i(v) = 0$ for all i, so $f_i(v) = 0$ for all $i \in J$, proving $W_S \subset \widetilde{W}_S$. If $v \in \widetilde{W}_S$, then $f_i(v) = 0$ for all $i \in J$, and for $i \notin J$, $\chi_i(x) - 1 = 0$, so $xv = v$ and $v \in W_S$, thus $\widetilde{W}_S \subseteq W_S$.

This business with W_S and $\widetilde{W}_S$ shows that W_S does not actually depend on S, but in fact only depends on J. Since J is a finite subset of $\{1, \ldots, n\}$ there are only finitely many such J, so there are only finitely many such $W_S = \text{fix}(S)$, so (1) is proved.

The proof of (2) is essentially the same idea as the proof of (1). We first address the case of singleton subsets of V. Let $v \in V$, and let

$$\begin{aligned} S_v &= \text{stab}_\text{p}(v) = \{x \in G : xv = v\} \\ J_v &= \{i : f_i(v) \neq 0\} \\ \widetilde{S}_v &= \{x \in G : \chi_i(x) = 0, \ \forall i \in J\} \end{aligned}$$

As in the proof of (1), one can verify that $S_v = \widetilde{S}_v$, so $\widetilde{S}_v$ depends only on J_v. As there are finitely many J_v, there are only finitely many S_v. If $W \subset V$ is any subset, then

$$W = \bigcap_{v \in W} \text{stab}_\text{p}(v)$$

so W is an intersection from a finite collection of sets, so there are only finitely many possible W.

Now we prove (3). As G is connected, it is irreducible. Consider

$$G_i = \{x \in G : \chi_i(x) \neq 0\} \subset G$$

This is an open subset, and if $\chi_i \neq 1$ then $G_i \neq \emptyset$. If the action of G on V is trivial, then the proposition is obvious, so we may assume the action is not trivial, so that some G_i is non empty. Let

$$J = \{i : G_i \neq \emptyset\} = \{i : \chi_i \neq 1\}$$

Since G is irreducible,

$$\bigcap_{i \in J} G_i \neq \emptyset$$

Also note that this set is dense in G, as it is a finite intersection of open sets. Let x be an element of the above intersection. We claim that such an x will work as in the statement of the proposition. If $xv = v$, then $(\chi_i(x) - 1)f_i(v) = 0$ for all i. Since $x \in G_i$, $\chi_i(x) - 1 \neq 0$, so $f_i(v) = 0$ for all $i \in J$. Thus for any $y \in G$,

$$f_i(v)(\chi_i(y) - 1) = 0 \qquad \forall i$$

so $yv = v$. As already noted, the set of such x is dense. □

Corollary 6.38. *Let H be a diagonalizable subgroup of an algebraic group G. Then*

1. *The sets*

$$\begin{aligned} &\{Z_G(S) : S \subset H\} \\ &\{Z_H(S) : S \subset G\} \end{aligned}$$

are finite.

2. *If H is connected, then there is a dense subset $S \subset H$ such that for any $x \in S$, $Z_G(H) = Z_G(x)$.*

Proof. (1) Apply the previous proposition in the case where H acts on G by conjugation. In this case,

$$Z_G(S) = \mathrm{fix}(S) \qquad Z_H(S) = \bigcap_{x \in S} \mathrm{stab}_{\mathrm{p}}(x)$$

so there are finitely many of each such set. (2) also follows from the previous proposition using the same action. ☐

7 Quotients and solvable groups

Theorem 7.1 (Lie-Kolchin theorem)**.** *Let V be an n-dimensional K-vector space, and let $G \subset \mathrm{GL}(V)$ be a closed subgroup which is connected and solvable. Then there is a basis of V such that G is contained in the subgroup of upper triangular matrices. That is, each element of G fixes a complete flag in V.*

Proof. Later in the course. ☐

Recall the notation

$$G_U = \{g \in G : g \text{ is unipotent}\}$$
$$G_S = \{g \in G : g \text{ is semisimple}\}$$

Theorem 7.2. *Let G be a connected solvable algebraic group. Then*

1. *G_U is a closed connected normal subgroup of G containing the commutator subgroup $[G, G]$.*

2. *If G is nilpotent, then G_S is a closed torus in G, and the morphism*

   $$G_S \times G_U \qquad (s, u) \mapsto su$$

   is an isomorphism of algebraic groups.

3. *The maximal (with respect to inclusion) tori are conjugates of each other. Also, if $T \subset G$ is a maximal torus, then G is a semidirect product of T and G_U, and the morphism*

   $$T \times G_U \to G \qquad (t, u) \mapsto tu$$

   is an isomorphism of varieties.

4. *If $S \subset G_S$ is any subset, then there is a torus containing S. In particular, S is abelian.*

5. *If $S \subset G_S$ is any subset, then $N_G(S) = Z_G(S)$ and $N_G(S)$ is connected.*

Proof. Some of this we will prove later in the course. ☐

Corollary 7.3. *Let G be a connected solvable algebraic group.*

1. *Every unipotent element (or unipotent subgroup) of G is contained in a connected unipotent group (namely G_U).*

2. *Every semisimple elemnt (or abelian subgroup of semisimple elements) of G is contained in a torus.*

Proof. Immediate from parts (1) and (4) of Theorem 7.2. □

Remark 7.4. In the previous corollary, (1) holds in more generality. In particular, the solvable hypothesis can be removed, although we may not get to proving this. However, the analogous statement for (2) does not hold, which is to say, there are counterexamples to the statement, "Every semisimple element in a connected algebraic group is contained in a torus." Again, we leave out the proof.

7.1 Complete varieties and Grassmannians

Remark 7.5. From now on in the course, the term "variety" will no longer refer to an affine variety, but instead to a quasi-projective variety. Roughly speaking, a quasi-projective variety is a (topological) space which has a finite open cover by affine varieties.

Really, a quasi-projective variety should be defined in terms of sheaves and schemes, but there's no time for that here. I apologize for being unable to give such a good intrinsic definition of quasi-projective variety as was given for abstract affine varieties. Our professor did not give a definition for this during the course. The reader may have to spend several hours reading up on algebraic geometry to really understand what is meant by "quasi-projective variety." As a silver lining, at that point you'll probably understand them better than I do.

Definition 7.6. A variety V is **complete** if for any variety W the projection map

$$V \times W \to W \qquad (v, w) \mapsto w$$

is a closed map.

Example 7.7. $\mathbb{A}^1$ is not complete. (Why? What choice of W fails to give a closed projection? I have no idea.)

Proposition 7.8 (Properties of complete varieites)**.** .

1. *A subvariety of a complete variety is complete.*

2. *The image of a complete variety under a morphism is complete.*

3. *A product of complete varieities is complete.*

4. *A complete affine variety consists of finitely many points.*

5. *A morphism from a connected complete variety to an affine variety is a constant map.*

6. *A projective variety is complete*

7. A product of projective varieties is projective, hence complete.

Definition 7.9. Let V be an $(n+1)$-dimensional K-vector space. The **projectivization** of V, denoted $\mathbb{P}(V)$, is the quotient of V by the action of $K^\times$. Alternatively, $\mathbb{P}(V)$ is the set of one dimensional subspaces of V. That is to say, for each $v \in V$, the element $[v] \in \mathbb{P}(V)$ is the equivalence class of all vectors λv for $\lambda \in K^\times$, or in the second interpretation, $[v]$ is the one dimensional subspace of V spanned by v.

If $v_1, \ldots, v_{n+1}$ is a basis of V, and $x \in V$ is written (uniquely) as

$$x = \sum_i x_i v_i$$

we call $(x_1, \ldots, x_{n+1})$ the coordinates of x in V, with respect to the basis $v_1, \ldots, v_d$. Similarly, we call $[x_1 : \cdots : x_{n+1}]$ the **homogeneous coordinates** of x in $\mathbb{P}(V)$, with respect to the basis $v_1, \ldots, v_d$. Note that homogeneous coordinates are not unique, in contrast with coordinates in V, since the same coordinates multiplied by a nonzero scalar represent the same point in $\mathbb{P}(V)$.

Definition 7.10. Let V be an $(n+1)$-dimensional K-vector space. Let

$$G_d(V) = \{U \subset V : U \text{ is a } d\text{-dimensional subspace}\}$$

$G_d(V)$ is called a **Grassmannian**, and often notated $G_d(n+1)$ or $\mathbb{G}(d-1, n)$. As we will see in a moment, $G_d(V)$ is a projective variety. It has dimension $d(n+1-d)$, though we omit justification of the dimension computation. To make $G_d(V)$ into a variety, we consider the map

$$G_d(V) \to \mathbb{P}\left(\bigwedge^d V\right) \qquad \operatorname{span}(v_1, \ldots, v_d) \mapsto [v_1 \wedge \cdots \wedge v_d]$$

First, we need to justify why this is well defined, since a priori another choice of basis for $\operatorname{span}(v_1, \ldots, v_n)$ might not give the same wedge product. However, if $v_1', \ldots, v_d'$ is another basis fo $\operatorname{span}(v_1, \ldots, v_d)$, then we have a change of basis matrix A which changes from the basis $v_1, \ldots, v_d$ to $v_1', \ldots, v_d'$, and one can check that

$$v_1' \wedge \cdots \wedge v_d' = (\det A)(v_1 \wedge \cdots \wedge v_d)$$

hence $[v_1' \wedge \cdots \wedge v_d'] = [v_1 \wedge \cdots \wedge v_d]$ in $\mathbb{P}(\bigwedge^d V)$. So the map is well defined. We also assert that the map is injective, and that the image is a closed subset of $\mathbb{P}(\bigwedge^d V)$, without proof. Hence $G_d(V)$ is identified with a subvariety of $\mathbb{P}(\bigwedge^d V)$, giving it the structure of a variety.

Example 7.11. We examine $G_2(4) = \mathbb{G}(1,3)$, which is the "smallest" example of a Grassmannian which is not just projective space. Just to be safe, we assume $\operatorname{char} K \neq 2$ for this example. Let v_1, v_2, v_3, v_4 be a basis of V, so

$$v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_2 \wedge v_3, v_2 \wedge v_4, v_3 \wedge v_4$$

is a basis of $\bigwedge^2 V$. So any $\omega \in \bigwedge^2 V$ can be written uniquely as

$$\omega = \sum_{i<j} a_{ij} v_i \wedge v_j \qquad a_{ij} \in K$$

So under the embedding $G_2(4) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$, the homogeneous coordinates of ω are

$$[a_{12} : a_{13} : a_{14} : a_{23} : a_{24} : a_{34}]$$

The image of $G_2(4)$ can be characterized as $\omega \in \mathbb{P}(\bigwedge^2 V)$ such that $\omega \wedge \omega = 0$ (some proof needed here, omitted). In terms of explicit coordinates, this condition is equivalent to

$$\begin{aligned}
\omega \wedge \omega = 0 &\iff \left(\sum a_{ij} v_i \wedge v_j\right) \wedge \left(\sum a_{ij} v_i \wedge v_j\right) = 0 \\
&\iff a_{12} v_1 \wedge v_2 \wedge a_{34} v_3 \wedge v_4 + a_{13} v_1 \wedge v_3 \wedge a_{24} v_2 \wedge v_4 + \cdots = 0 \\
&\iff 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) v_1 \wedge v_2 \wedge v_3 \wedge v_4 = 0 \\
&\iff 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) = 0
\end{aligned}$$

Since we assumed char $\neq 2$, this is equivalent to

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$$

All this to say, $G_2(4)$ is the solution to a quadratic homogeneous polynomial in $\mathbb{P}^5$, so it is what is known as a quadric surface.

7.2 Flag varieties

Definition 7.12. Let V be an n-dimensional K-vector space. We define $\mathcal{F}(V)$ to be the set of complete flags in V.

$$\mathcal{F}(V) = \{0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V\}$$

This object is called a **flag variety**. The next proposition justifies calling it a variety.

Proposition 7.13. *The map*

$$\mathcal{F}(V) \to G_0(V) \times \cdots \times G_n(V) \qquad V_0 \subset \cdots \subset V_n \mapsto (V_0, \ldots, V_n)$$

is an embedding with closed image.

Remark 7.14. Using the previous proposition, $\mathcal{F}(V)$ is given the structure of a projective variety.

Remark 7.15. This is probably beyond our class, but GL_n acts transitively on $\mathcal{F}(V)$ as an algebraic group, and the stabilizer of any given flag is a subgroup B which is conjugate to upper triangular matrices in GL_n. Such a subgroup is called a **Borel subgroup**. This also implies that as a set, $\mathcal{F}(V) \cong \mathrm{GL}_n / B$.

7.3 Quotients

We begin by recalling the universal property of the quotient in the category of groups.

Proposition 7.16. *Let G be a group and $H \subset G$ a normal subgroup. Let $\pi : G \to G/H$ be the quotient map $g \mapsto gH$. If $\alpha : G \to G'$ is a group homomorphism and $H \subset \ker \alpha$, then there exists a unique group homomorphism $\overline{\alpha} : G/H \to G'$ making the following diagram commute.*

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \overline{\alpha}} & \\ G/H & & \end{array}$$

Our next goal is to discuss the quotient of an algebraic group G by a closed subgroup H. Ideally, this will result in an object denoted G/H, which is an algebraic group, with a quotient map $\pi : G \to G/H$. Notice that we aren't going to be restricted to the case where H is a normal subgroup, which immediately alerts us to the fact that G/H will probably not be a group, in general, but will usually only be a variety.

One issue is to decide what G/H should be as a set, although this is not terribly difficult. If it is going to behave like a quotient, it should probably be something akin to a coset space. If we aren't going to require it to be a group, then it should at least be a variety, since we're interested in the category of algebraic groups.

The larger issue is to decide what is meant by "quotient map" $\pi : G \to G/H$. As noted above, G/H should have the structure of a variety, so at the very least π should be a morphism of varieties. Usually quotients maps are surjective, so that should probably be a requirement. One way to ensure that G/H is a "coset space" is to require that fibers of π are cosets of H, or equivalently that π is constant on cosets of H, so that seems reasonable.

The gold standard for any categorical type construction is to have a universal property, so the biggest goal of defining a quotient map should be that it results in a universal property akin to Proposition 7.16.

Here is a botched first attempt at an "obvious" definition.

Definition 7.17 (Fake definition)**.** Let H be a closed normal subgroup of an algebraic group G. The **quotient** of G by H is the quotient in the category of groups (denoted G/H), and it has the structure of an algebraic group via... something.

For whatever reason, this is not a satsifactory definition. The first reason is somewhat obvious - it is not so clear how to make G/H into a variety, and we would need to do so in a way that is compatible with the group structure.

Basically, this issue is that while quotients in the category of groups are well-behaved (we know what conditions on H are needed for it to exist), but in the category of varieties quotients are not so well-behaved. That is to say, sometimes the set which is the obvious choice for a quotient of varieties is not an object in the category of varieties.

Given all this discussion, the subtleties of the next definition are somewhat more expected.

Definition 7.18 (Real definition)**.** Let G be an algebraic group and $H \subset G$ a closed subgroup. A pair (π, V) where V is a variety and $\pi : G \to V$ is a morphism of varieties is a **quotient** of G by H if

1. The fibers of π are left[5] cosets of H in G. In particular, π is surjective.

[5] Right cosets may also be used, not a big difference.

2. π is an open map.

3. If $U \subset V$ is open, then
$$\pi^*(K[U]) = K[\pi^{-1}(U)]^H$$

Note that π has most of the properties we mentioned - it is constant on cosets of H, it is surjective, and it is a morphism of varieties. The universal property remains to be seen.

Remark 7.19. The last condition is somewhat mysterious, so we spell it out more. H acts on G by left multiplication,

$$H \times G \to G \qquad (h, g) \mapsto h \cdot g = hg$$

so H also acts on $K[G]$,

$$H \times K[G] \to K[G] \qquad (h, f) \mapsto h \cdot f = \rho_h^* f$$

Now consider $U \subset V$. Since the fibers of π are cosets of H, $\pi^{-1}(U)$ is a union of left cosets of H. We want to say that $K[\pi^{-1}(U)]$ is an H-invariant subspace of $K[G]$. We need to check that if $f \in K[\pi^{-1}(U)]$ then $h \cdot f \in K[\pi^{-1}(U)]$. For $x \in \pi^{-1}(U)$,

$$h \cdot f(x) = \rho_h^* f(x) = f(xh)$$

Since x, xh are in the same coset of H and $\pi^{-1}(U)$ is a union of left cosets of H, $x \in \pi^{-1}(U)$ implies $xh \in \pi^{-1}(U)$, so $h \cdot f \in K[\pi^{-1}(U)]$. Thus $K[\pi^{-1}(U)]$ is an H-invariant subspace, meaning it makes sense to talk about an action of H on $K[\pi^{-1}(U)]$.

$$H \times K[\pi^{-1}(U)] \times K[\pi^{-1}(U)] \qquad (h, f) \mapsto h \cdot f = \rho_h^* f$$

Hence it also makes sense to talk about $K[\pi^{-1}(U)]^H$, as in the previous definition.

Below, we also depict some of this with commutative diagrams below. The maps res denote restriction, for example, if $f \in K[V]$ then $\text{res}(f) = f|_U$.

$$\begin{array}{ccc} G & \xrightarrow{\pi} & V \\ \uparrow & & \uparrow \\ \pi^{-1}(U) & \xrightarrow{\pi} & U \end{array} \qquad \begin{array}{ccc} K[G] & \xleftarrow{\pi^*} & K[V] \\ \downarrow \text{res} & & \downarrow \text{res} \\ K[\pi^{-1}(U)] & \xleftarrow{\pi^*} & K[U] \end{array}$$

From this picture, we see that $\pi^*(K[U])$ maps to some subset of $K[\pi^{-1}(U)]$, which is a priori not the same as $K\pi^{-1}(U)]^H$. If π is a quotient map, then these things must be the same.

Example 7.20. If G_1, G_2 are algebraic groups, then the projection

$$\pi : G_1 \times G_2 \to G_1 \qquad (x_1, x_2) \mapsto x_1$$

is a quotient of $G_1 \times G_2$ by G_2 (by G_2 as a subgroup we obviously mean $1 \times G_2$). For the purpose of familiarizing ourselves with the definition, let's verify this, especially focusing on condition 3. It is clear that the fibers are cosets of $1 \times G_2$. If $A \subset G_1 \times G_2$ is open, then $(G_1 \times G_2) \setminus A$ is closed. Also, for any $y \in G_2$, $G_1 \times y \subset G_1 \times G_2$ is closed, and

$$G_1 \setminus \pi(A) = \bigcap_{y \in G_2} \pi\left((G_1 \times y) \setminus A\right)$$

The right hand side is an intersection of closet sets so it is closed, so $\pi(A)$ is closed in G_1. Now for the mysterious 3rd condition. Let $U \subset G_1$ be open. We need to check that $\pi^* K[U] = K[\pi^{-1}(U)]^{1 \times G_2}$.

$$\begin{array}{ccc} G & \xrightarrow{\pi} & V \\ \uparrow & & \uparrow \\ \pi^{-1}(U) & \xrightarrow{\pi} & U \end{array} \qquad\qquad \begin{array}{ccc} K[G] & \xleftarrow[\pi^*]{} & K[V] \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} \\ K[\pi^{-1}(U)] & \xleftarrow[\pi^*]{} & K[U] \end{array}$$

The subgroup $1 \times G_2$ acts on $K[\pi^{-1}(U)]$ as follows. For $\alpha \in K[\pi^{-1}(U)]$ and $(x, y) \in \pi^{-1}(U)$ and $z \in G_2$,

$$((1, z) \cdot \alpha)(x, y) = \alpha(x, yz)$$

If $f \in K[U]$ then $\pi^f(x, y) = f(x)$ for all $(x, y) \in G_1 \times G_2$. So

$$((1, z) \cdot \pi^* f)(x, y) = f(x, yz) = f(x) = \pi^* f(x, y)$$

Thus $(1, z) \cdot \pi^* f = \pi^* f$, so $\pi^* f \in K[\pi^{-1}(U)]^{1 \times G_2}$, so $\pi^* K[U] \subset K[\pi^{-1}(U)]^{1 \times G_2}$. For the reverse inclusion, let $\alpha \in K[\pi^{-1}(U)]^{1 \times G_2}$. Then for all $z \in G_2$, $(x, y) \in \pi^{-1}(U)$,

$$(1, z) \cdot \alpha = \alpha \implies ((1, z) \cdot \alpha)(x, y) = \alpha(x, yz) = \alpha(x, y)$$

Hence $\widetilde{\alpha} : U \to K$ given by $\widetilde{\alpha}(x) = \alpha(x, y)$ is independent of y, and $\pi^* \widetilde{\alpha} = \alpha$, so $\alpha \in \pi^* K[U]$. This establishes the opposite inclusion we needed, so $\pi^* K[U] = K[\pi^{-1}(U)]^{1 \times G_2}$.

Example 7.21. A flag variety $\mathcal{F}(V)$ is the quotient of GL_n by a Borel subgroup B.

Proposition 7.22 (Universal property of quotients)**.** *If (π, V) is a quotient of G by H, and $\alpha : G \to G'$ is a homomorphism of algebraic groups such that α is constant on cosets of H (equivalently, $H \subset \ker \alpha$), then there is a unique morphism of varieties $\overline{\alpha} : V \to G'$ making the following diagram commute.*

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \overline{\alpha}} & \\ V & & \end{array}$$

Proof. (Sketch) There is only one reasonable way to construct $\overline{\alpha}$ so that the diagram commutes. Since π is surjective, it suffices to define $\overline{\alpha}$ on the image of π.

$$\overline{\alpha}(\pi(g)) = \alpha(g)$$

Since the fiber of $\pi(g)$ is the coset gH and $\alpha(gH) = \alpha(g)$, this is well defined. Uniqueness of $\overline{\alpha}$ is clear. The difficulty is now to show that $\overline{\alpha}$ is a morphism of varieties. I don't know how to do this, so I'll just leave this here. $\square$

Remark 7.23. Assuming the universal property above, if a quotient exists, then clearly the object V is unique up to isomorphism and the morphism π is also unique up to isomorphism (of morphisms, meaning a certain commutative triangle).

Definition 7.24. Let G be an affine algebraic group, acting on a variety V (as an algebraic group). For $v \in V$, let

$$G_v = \operatorname{stab}(v) = \{g \in G : gv = v\}$$

The **orbit map** associated to v is the morphism of varieties

$$\pi : G \to V \qquad g \mapsto gv$$

Notice that (π, V) has some of the properties required of a quotient of G by G_v. The fibers are cosets of G_v, and π is an open map, although it takes some proving. The last property, that

$$\pi^* K[U] = K[\pi^{-1}(U)]^{G_v}$$

does not hold in general, but does "usually" hold. Our next goal is to make this more precise, by spelling out an equivalent condition in terms of tangent spaces.

Remark 7.25. Let $\alpha : U \to V$ be a morphism of varieties. Recall that for $u \in U$, we define the tangent space $T_u U$ to be the set of derivations of $K[U]$ at u, and that the differential of α at u is given by

$$(d\alpha)_u : T_u U \to T_{f(u)} V \qquad (d\alpha)_u(D)(f) = D(f \circ \alpha)$$

Proposition 7.26. *Let G be an algebraic group acting on a variety V. Fix $v \in V$, and let $\pi : G \to G_v$ be the orbit map. Then (π, V) is a quotient of G by G_v if and only if the differential*

$$(d\pi)_1 : T_1 G \to T_v V$$

[6] *is surjective. In particular, if $\operatorname{char} K = 0$, this is always surjective.*

Theorem 7.27 (Existence of quotients)**.** *Let G be an affine algebraic group and $H \subset G$ a closed subgroup. Then the quotient of G by H exists. If H is normal, then G/H is an affine algebraic group.*

Remark 7.28. We give no proof of the previous theorem, but we can say a little bit about the method of proof. The quotient G/H is realized via an orbit map. The trick is to choose a finite dimensional vector space V which has a G-action so that H is the stabilizer of some $v \in V$. Then consider the orbit map $\pi : G \to V, g \mapsto gv$, and show that (π, V) is a quotient of G by H. The proof is very heavy in algebraic geometry.

Lemma 7.29 (Compatibility of quotient with product)**.** *Let G, G' be algebraic groups with $H \subset G$ a closed subgroup. Then*

$$\frac{G \times G'}{H \times 1} \cong \frac{G}{H} \times G'$$

[6] 1 refers to the identity element of G.

Proof. Let $\pi : G \times G' \to \frac{G \times G'}{(H \times 1}$ be the quotient map and $\eta : G \to \frac{G}{H}$ be the quotient map. Let $\iota : G \times 1 \to G \times G'$ be the inclusion.

$$\begin{array}{ccc} G \times 1 & \xhookrightarrow{\iota} & G \times G' \\ \big\downarrow{\scriptstyle \eta \times 1} & & \big\downarrow{\scriptstyle \pi} \\ \frac{G}{H} \times 1 & & \frac{G \times G'}{H \times 1} \end{array}$$

The map π is constant on cosets of $H \times 1$, so $\iota \circ \pi$ is constant on cosets of $H \times 1$, so by the universal property of quotients there is a unique morphism $\alpha : \frac{G}{H} \times 1 \to \frac{G \times G'}{H \times 1}$ making the square commute. Similarly, by the universal property, there is a unique morphism β going the other way to make the square commute.

$$\begin{array}{ccc} G \times 1 & \xhookrightarrow{\iota} & G \times G' \\ \big\downarrow{\scriptstyle \eta \times 1} & & \big\downarrow{\scriptstyle \pi} \\ \frac{G}{H} \times 1 & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & \frac{G \times G'}{H \times 1} \end{array}$$

By uniqueness of α, β, they must be inverses, so we obtain the desired isomorphism. □

7.4 Borel fixed point theorem

This section is all about the long proof of the Borel fixed point theorem, and several powerful corollaries. First we need an algebraic geometry lemma.

Lemma 7.30. *Let G be a connected affine algebraic group acting transitively on homogeneous varieties V_1, V_2.* [7] *Let $f : V_1 \to V_2$ be a morphism of G-varieties. Suppose f has finite fibers. If V_2 is complete, then V_1 is also complete.*

Proof. Omitted. □

Theorem 7.31 (Borel fixed point theorem)**.** *If G is a connected, solvable, algebraic group acting on a complete variety V, then there is a fixed point for the action.*

Proof. We induct on the dimension of G. The case $\dim G = 0$ is clear, since in this case G is trivial, since it is connected. Now consider $\dim G > 0$.

Since G is solvable, the commutator subgroup $DG = [G, G]$ is a proper subgroup. It is a closed subgroug, so $\dim DG < \dim G$. Also, it is connected and solvable. Thus by induction hypothesis, there is a fixed point. That is, the following set is nonempty.

$$W = V^{DG} = \{v \in V : gv = v, \ \forall g \in DG\}$$

We want to show that W is a subvariety of V. Consider the product variety $DG \times V \times V$ and its closed subvarieties

$$A = \{(g, v, gv) : g \in DG\} \qquad B = \{(g, v, v) : g \in DG\}$$

[7]Homogeneous means that V_1, V_2 are roughly quotients of G.

Since A, B are closed, $A \cap B$ is closed, and W is the image of $A \cap B$ under the projection

$$DG \times V \times V \to V \qquad (g, v, w) \mapsto w$$

Since this projection is a closed map, W is closed, so it is a subvariety of V. Now we claim that W is (setwise) invariant under the action of G. For $h \in DG, g \in G, w \in W$, we have

$$h(gw) = gg^{-1}h(gw)$$

Since DG is a normal subgroup, $g^{-1}hg \in DG$, so $g^{-1}hgw = w$, so

$$h(gw) = g(g^{-1}hg)w = gw$$

Thus $gw \in W$, so W is (setwise) G-invariant. Now watch carefully, because the next step is a somewhat confusing reduction/replacement. Let $\widetilde{W} \subset W$ be the smallest dimensional orbit of G acting on W, so that $\widetilde{W}$ is closed. Now G is acting on $\widetilde{W}$, and the DG-action on $\widetilde{W}$ is trivial. We now replace our original V by this $\widetilde{W}$, and just seek to find a fixed point of G in $\widetilde{W}$. This will be a fixed point in V, which is what we need to complete the proof.

So the situation now is: G acts on a variety V, and DG acts trivially on V, and we want a fixed point for G to complete the induction. Let $v \in V$ and $G_v = \operatorname{stab}(v) = \{g \in G : gv = v\}$, and consider the orbit map

$$\eta : G \to V \qquad g \mapsto gv$$

This is constant on cosets of G_v in G. By existence of quotients, there is a quotient G/G_v (more properly denoted $(\pi, G/G_v)$ and by the universal property there is an isomorphism of varieties

$$\phi : G/G_v \to V \qquad gG_v \mapsto gv$$

Since G_v is normal in G, G/G_v is an affine algebraic group. Now we use Lemma 7.30. Since ϕ is a bijection and V is complete, G/G_v is also complete by the lemma.

Since G/G_v is an affine algebraic group, being complete means it is a finite set of points, and since it is connected, it is a single point. Thus $G/G_v = \{e\}$ so $G = G_v$. That is to say, v is a fixed point of G. This completes the induction, and hence the proof. □

One consequence of the Borel fixed point theorem is that we can now prove the Lie-Kolchin theorem, which we stated earlier.

Theorem 7.32 (Lie-Kolchin)**.** *Let G be a connected solvable closed subgroup of* $\mathrm{GL}(V)$*. Then there is a complete flag in V which is fixed by G.*

Proof. G acts on $\mathcal{F}(V)$ and $\mathcal{F}(V)$ is complete, so by the Borel fixed point thereom there is a fixed flag. □

8 Borel subgroups

Definition 8.1. Let G be a connected affine algebraic group. A **Borel subgroup** is a maximal connected solvable closed subgroup of G. (By maximal, we mean maximal with respect to inclusion among connected solvable subgroups of G.)

Remark 8.2. Every connected algebraic group has a Borel subgroup, just using maximality and dimension considerations.

Example 8.3. Let $G = \mathrm{GL}(V)$, and let B be the subgroup of upper triangular matrices. B is the prototypical example of a Borel subgroup.

Remark 8.4. Any two Borel subgroups B, B' of $\mathrm{GL}(V)$ are conjugate, just by a change of basis matrix. This follows from the Lie-Kolchin theorem.

To explain this a bit more, choose a complete flag in V which B fixes (using Lie-Kolchin). So with respect to the basis defined by this flag, B is the subgroup of upper triangular matrices. Choose another flag which B' fixes, so B' is upper triangular matrices in a different basis. Then conjugating G by the change of basis matrix sends B to B'. By maximality, the conjugate of B is all of B'.

Our next goal is to extend the previous remark to a general algebraic group G. That is, we want to show that any two Borel subgroups are conjugate. Without thinking too hard, one wonders if this might be easy, since G can be embedded in some $\mathrm{GL}(V)$, and the Borel subgroups B, B' can be embedded into the upper triangular subgroup in differing bases of V. However, it is not clear how to proceed form here, and the argument halts.

Lemma 8.5. *Let G be a connected affine algebraic group and B a Borel subgroup of maximal dimension. Then there is a representation*

$$\rho : G \to \mathrm{GL}(V)$$

so that there is a one-dimensional subspace $W_1 = \mathrm{span}(w_1) \subset V$ with $B = \mathrm{stab}(W_1)$, and so that

$$G/B \to Gw_1 = \mathrm{orb}(w_1) \qquad xB \mapsto xw_1$$

is an isomorphism of varieties.

Proof. Only a somewhat loose sketch of a proof was given in class, so I have left it out. □

Theorem 8.6. *Let G be a connected affine algebraic group.*

1. *If B is a Borel subgroup, then G/B is projective, and hence complete.*
2. *Any two Borel subgroups of G are conjugate. That is, for Borel subgroups B, B', there exists $g \in G$ such that $gBg^{-1} = B'$,*

Proof. Done it class but very confusing. Maybe I'll get back to this some day. □

Corollary 8.7. *Let P be a closed subgroup of a connected affine algebraic group G. The following are equivalent.*

1. *The quotient G/P is a complete variety.*
2. *P contains a Borel subgroup.*

Proof. (1) $\implies$ (2) Let B be a Borel subgroup of G. G acts on G/P by left translation, and we restrict this action to B.

$$B \times G/P \to G/P \qquad (b, gP) \mapsto bgP$$

Since G/P is complete, we can use the Borel fixed point theorem 7.31 to obtain a fixed point gP, so for all $b \in B$,

$$bgP = gP \implies g^{-1}bgP = P \implies g^{-1}bg \in P \implies g^{-1}Bg \subset P$$

Since $g^{-1}Bg$ is a conjugate of B, it is a Borel subgroup, hence P contains a Borel subgroup.

(2) $\implies$ (1) Let $B \subset P$ be a Borel subgroup. Let $\eta : G \to G/P$ and $\pi : G \to G/B$ be the respective quotient maps. Since η is constant on cosets of P it is also constant on cosets of B. So by the universal property of quotients, we get a morphism $G/B \to G/P$ fitting into the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta} & G/P \\ \downarrow{\scriptstyle \pi} & \overset{\overline{\pi}}{\nearrow} & \\ G/B & & \end{array}$$

Since η is surjective, $\overline{\pi}$ must also be surjective. Since B is a Borel subgroup, G/B is complete, and then because G/P is a homomorphic image of a complete variety, it is also complete. □

Definition 8.8. A subgroup P of a an algebraic group G satsifying the equivalent conditions above is a **parabolic subgroup**. Note that it is immediate that a minimal parabolic subgroup is a Borel subgroup.

Recall that a torus (plural tori) in an algebraic group is a subgroup T which is diagonalizable and connected.

Corollary 8.9. *Let G be a connected affine algebraic group.*

1. *Any two maximal tori in G are conjugate.*

2. *Any two maximal connected unipotent subgroups of G are conjugate.*

Proof. (1) Let T, T' be maximal tori, so they are abelian, hence solvable. So they are each contained in some Borel subgroup, that is, there are Borel subgroups B, B' of G such that $T \subset B, T' \subset B'$. We know B, B' are conjugate, so B contains a conjugate of T'. So may as well assume $T, T' \subset B$, by replacing T' with its conjugate in B. Now by part (3) of Theorem 7.2, since T, T' are now maximal tori inside a solvable connected algebraic group, they are conjugate.

(2) Let U be a maximal connected unipotent subgroup, so U is solvable. So U is contained in some Borel subgroup B. In particular, $U \subset B_U$, that is, U is contained in the unipotent subgroup of B. By maximality of U, $U = B_U$. If U' is another maximal connected unipotent subgroup with $U' \subset B'$, then $U' = B'_U$. Since B, B' are conjugate, and conjugating takes unipotent elements to unipotent elements, the same conjugation takes $U = B_U$ to $U' = B'_U$. □

Corollary 8.10. *Let G be a connected affine algebraic group.*

1. *If $\alpha : G \to G$ is a homomorphism of algebraic groups such that $\alpha|_B = \mathrm{Id}_B$ for some Borel subgroup B, then $\alpha = \mathrm{Id}_G$.*

2. *$Z_G(B) \subset Z_G(G)$, hence $Z_G(B) = Z_G(G)$.*

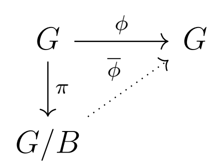
Proof. (1) Consider the morphism of varieties (not generally a group homomorphism)

$$\phi : G \to G \qquad x \mapsto \alpha(x)x^{-1}$$

We claim that ϕ is constant on cosets of B. Let gB be a coset, and let $g, g' \in gB$, so $g' = gb$ for some $b \in B$. Then

$$\phi(g') = \alpha(gb)(gb)^{-1} = \alpha(g)\alpha(b)b^{-1}g^{-1} = \alpha(g)bb^{-1}g^{-1} = \alpha(g)g^{-1} = \phi(g)$$

So ϕ is constant on cosets of B. Thus by the universal property of quotients ϕ factors through the quotient G/B in the following commutative triangle.



Since G is connected, G/B is also connected, hence irreducible. Since G/B is also complete and G is affine, $\overline{\phi}$ is constant, hence $\mathrm{im}\,\overline{\phi} = \mathrm{im}\,\phi = \{e\}$. Hence $\alpha = \mathrm{Id}_G$.

(2) Note that the inclusion $Z_G(G) \subset Z_G(B)$ is always true for any subgroup B of a group G. If $x \in Z_G(B)$, apply (1) to the morphism

$$\alpha : G \to G \qquad g \mapsto xgx^{-1}$$

Then $\alpha|_B = \mathrm{Id}$ by choice of x, hence by (1) $\alpha = \mathrm{Id}|_G$, which is to say, $x \in Z_G(G)$. □

Remark 8.11. A similar property to the previous corollary holds more generally for group actions on affine varieties. That is, if a connected affine algebraic group G acts on an affine variety V, and $v \in V$ is fixed by a Borel subgroup B, then v is fixed by all of G. The proof of this is similar to the proof of the previous corollary.

Lemma 8.12. *Let N be a nontrivial nilpotent group. Then $Z(N)$ is nontrivial.*

Proof. This is a standard fact, but the proof is short so we include it here. Let $N^0 = N, N^1 = [N, N]$, $N^{k+1} = [N^k, N]$. Since N is nilpotent, there exists n so that N^n is trivial. Choose n so that N^n is trivial but N^{n-1} is not trivial. Then because $N^n = [N^{n-1}, N]$ is trivial, for all $x \in N^{n-1}$ and all $y \in N$,

$$xyx^{-1}y^{-1} = 1 \implies xy = yx \implies x \in Z(N) \implies N^{n-1} \subset Z(N)$$

Since N^{n-1} was chosen to be nontrivial, $Z(N)$ is nontrivial. □

Corollary 8.13. *Let G be a connected affine algebraic group. If a Borel subgroup B is nilpotent, then $G = B$ (hence G is nilpotent).*

Proof. We proceed by induction on the dimension of B. If $\dim B = 0$, then since B is connected it is a point, and $G/B \cong G$. Since G/B is complete and irreducible and G is affine, this forces $G = \{e\} = B$.

Now consider $\dim B \geq 1$. Since B is nilpotent, it has nontrivial center, so there is a nontrivial closed subgroup $C \subset Z(B)$ with $\dim C \geq 1$. By the previous corollary, $C \subset Z(G)$, in particular, C is a normal subgroup, so G/C is a connected affine algebraic group. We claim that B/C is a Borel subgroup of G/C. Consider the composition of quotient maps

$$G \to G/C \to \frac{G/C}{B/C}$$

Let $\eta : G \to \frac{G/C}{B/C}$ denote this composition. Then η is constant on cosets of B, so by the universal property of quotients, it factors through G/B as in the following diagram.

$$\begin{array}{ccc} G & \xrightarrow{\eta} & \frac{G/C}{B/C} \\ \downarrow & \overset{\overline{\eta}}{\nearrow} & \\ G/B & & \end{array}$$

Since B is a Borel subgroup G/B is complete, and since $\frac{G/C}{B/C}$ is a homomorphic image of a complete variety, it is complete. We know that B/C is connected and solvable, so by Corollary 8.7, B/C contains a Borel subgroup of G/C. Since it is already connected and solvable, by maximality it is a Borel subgroup of G/C.

Now we can complete the induction. We have $\dim B/C < \dim B$ and B/C nilpotent, so by induction hypothesis, $B/C = G/C$ hence $B = G$. □

8.1 Cartan subgroups

Throughout this, we need G to be a connected algebraic group. If we forget to mention that G is connected at any point, it is relatively safe to assume that it is included as a hypothesis.

Definition 8.14. Let G be a connected algebraic group. A closed subgroup $C \subset G$ is a **Cartan subgroup** if $C = Z_G(T)^0$ for some maximal torus T.

Remark 8.15. We claim any two Cartan subgroups of G are conjugate. Let $C = Z_G(T)^0, C' = Z_G(T')^0$ for two maximal tori T, T'. By Corollary 8.9, T, T' are conjugate, $T' = gTg^{-1}$ for some $g \in G$. Then

$$\begin{aligned} Z_G(T') &= \left\{x \in G : xt'x^{-1} = t',\ \forall t' \in T'\right\} \\ &= \left\{x \in G : xgtg^{-1}x^{-1} = gtg^{-1},\ \forall t \in T\right\} \\ &= \left\{x \in G : (g^{-1}xg)t(g^{-1}x^{-1}g) = t,\ \forall t \in T\right\} \\ &= \left\{x \in G : (g^{-1}xg)t(g^{-1}xg)^{-1} = t,\ \forall t \in T\right\} \\ &= gZ_G(T)g^{-1} \end{aligned}$$

Thus

$$C' = Z_G(T')^0 = (gZ_G(T)g^{-1})^0 = gZ_G(T)^0g^{-1} = gCg^{-1}$$

so C, C' are conjugate.

Proposition 8.16. *Let G be a connected affine algebraic group and $C = Z_G(T)^0$ a Cartan subgroup where T is a maximal torus. Then T is the unique maximal torus in C, hence T is the unique maximal torus so that $C = Z_G(T)^0$.*

Proof. It is clear that T is a maximal torus in C, and that T is normal in C. Then because T is normal in C, the only conjugate of T in C is itself. Since all maximal tori in C are conjugate, this shows T is the unique maximal torus in C. □

Corollary 8.17. *Let G be a connected affine algebraic group and C a Cartan subgroup. Then C is nilpotent.*

Proof. Let $C = Z_G(T)^0$ be a Cartan subgroup with T a maximal torus. Choose a Borel subgroup B of C with

$$T \subset B \subset C = Z_G(T)^0$$

This exists because by definition B is a maximal connected solvable group, and T is a connected solvable subgroup. By part (3) of Theorem 7.2, B is a semidirect product of T and B_U. We claim that it is in fact a direct product. To show this, it suffices to show that $T \cap B_U = \{e\}$. But this is clear because every element of T is semisimple, and every element of B_U is unipotent, and the only element of an algebraic group which is both semisimple and unipotent is the identity. Thus $B = T \times B_U$.

Since T is abelian, it is nilpotent, and B_U is also nilpotent. Hence B is nilpotent, so by Corollary 8.13, $B = C$ hence C is nilpotent. □

Lemma 8.18. *Let C be a Cartan subgroup of a connected affine algebraic group G. Then there exists $t \in C$ such that*

$$\left\{xC \in G/C : x^{-1}tx \in C \text{ is finite}\right\}$$

Proof. Let $C = Z_G(T)^0$ with T a maximal torus. By Corollary 6.38, there exists $t \in T$ such that $Z_G(t) = Z_G(T)$. We will eventually show that this is the required t.

If $C' = Z_G(T')^0$ is any Cartan subgroup containing t, then $t \in C' = Z_G(T')^0$ hence $T' \subset Z_G(t) = Z_G(T)$. Since T' is connected, $' \subset Z_G(T)^0 = C$. Hence by uniqueness (Proposition 8.16), $T = T', C = C'$. All this to say, C is the unique Cartan subgroup containin t.

Now for $x \in Z_G(t) = Z_G(T)$ we have $x^{-1}tx \in C$ hence $t \in xCx^{-1}$. Since xCx^{-1} is also a Cartan subgroup containing t, by uniqueness $xCx^{-1} = C$. Thus $x \in N_G(C) \subset N_G(T)$. By part (5) of Theorem 7.2, $N_G(T)^0 = Z_G(T)^0 = C$. That is, if $x \in N_G(T)^0$ then $xC = C$, so the number of distinct cosets xC with $x \in vtx \in C$ is bounded above by the size of $N_G(T)/N_G(T)^0$. Since this latter quantity is finite, the number of such xC is finite, as claimed. □

Remark 8.19. The following is a minor fact in point-set topology which we will need in the next proof. If $\eta : X \to Y$ is a surjective open map, and $Z \subset Y$ such that $\eta^{-1}(Z)$ is closed, then Z is closed. We give a picture, then some more explanation.

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Y \\ \uparrow & & \uparrow \\ \eta^{-1}(Z) & \xrightarrow{\eta} & Z \end{array}$$

Because $\eta^{-1}(Z)$ is closed, its complement $X \setminus \eta^{-1}(Z)$ is open. Then because η is surjective,

$$\eta(X \setminus \eta^{-1}(Z)) = Y \setminus Z$$

Since η is an open map, this shows $Y \setminus Z$ is open, hence Z is closed.

The next result was originally viewed as a major result, though now it is called a lemma. In any case, the proof is long and arduous.

Lemma 8.20 (Density lemma/Borel density theorem)**.** *Let G be a connected affine algebraic group. The union of all Cartan subgroups of G contains a dense open subset of G.*

Proof. Fix a Cartan subgroup $C = Z_G(T)^0$ with T a maximal torus. Since all Cartan subgroups are conjugate, we need to show that

$$K = \bigcup_{g \in G} gCg^{-1}$$

contains an open dense subset of G. Consider

$$S_0 = \{(x, y) \in G \times G : x^{-1}yx \in C\}$$

Note that S_0 is closed in $G \times G$ because it is the preimage of C under the morphism (of varieties)

$$G \times G \to G \qquad (x, y) \mapsto x^{-1}yx$$

We also claim S_0 is irreducible. To see this, consider the morphism

$$\theta : G \times G \to G \times G \qquad (x, y) \mapsto (x, xyx^{-1})$$

Then observe

$$\begin{aligned} \theta(G \times C) &= \{(x, xyx^{-1}) \in G \times G : x \in G, y \in C\} \\ &= \{(x, z) \in G \times G : x \in zx \in C, x \in G\} \\ &= S_0 \end{aligned}$$

Since G, C are connected, $G \times C$ is connected, hence irreducible. Then since S_0 is the image of $G \times C$ under a morphism of varieties, S_0 is also irreducible. We also note that S_0 is a union of some left cosets of $C \times 1$ in $G \times G$. That is, if $(x, y) \in S_0$, then $(xc, y) \in S_0$, since

$$x \in yx \in C \implies (xc)^{-1}yxc = c^{-1}x^{-1}yxc \in C$$

By Lemma 7.29, quotients and products are compatible, which in our case we apply to obtain an isomorphism

$$\frac{G \times G}{C \times 1} \cong \frac{G}{C} \times G$$

In particular, the following is a quotient map.

$$\eta : G \times G \to G/C \times G \qquad (g, g') \mapsto (gC, g')$$

So η is a surjective open map. Now we apply Remark 8.19 in the case $Z = \eta(S_0)$. In this instance, $\eta^{-1}(Z) = \eta^{-1}(\eta(S_0))$ is closed as it is a union of coset/translations of S_0, which is closed. So by the remark, $\eta(S_0)$ is closed in $\frac{G}{C} \times G$. Let $S = \eta(S_0)$.

$$S = \left\{(xC, y) \in G/C \times G : x^{-1}yx \in C\right\}$$

Now consider the two projections

$$\begin{aligned} p_1 &: S \to G/C \qquad (xC, y) \mapsto xC \\ p_2 &: S \to G \qquad\quad (xC, y) \mapsto y \end{aligned}$$

We finally have some justification for why this set S might be related to the original claim, because

$$p_2(S) = K = \bigcup_{g \in G} gCg^{-1}$$

We want to show that $p_2(S)$ contains a dense open subset of G, though first we need to establish some dimension facts using p_1. It is clear that $p_1 : S \to G/G$ is surjective, since $p_1(xC, 1) = xC$. Also, the fiber over x_0C is

$$\begin{aligned} p_1^{-1}(x_0C) &= \left\{(x_0C, y) \in G/C \times G : x_0^{-1}yx_0 \in C\right\} \\ &= \left\{(x_0C, y) \in G/C \times G : y \in x_0Cx_0^{-1}\right\} \\ &\cong x_0Cx_0^{-1} \end{aligned}$$

So each fiber has equal dimension, equal to the dimension of C. Using a standard result about fiber dimensions for dominant morphisms of varieties,

$$\dim C = \text{fiber dimension} = \dim S - \dim p_1(S) = \dim S - \dim G/C = \dim S - (\dim G - \dim C)$$

Rearranging this easily gives $\dim S = \dim G$. This finishes our use of p_1, now we consider $p_2 : S \to \overline{p_2(S)}$. Every variety is thick in itself, so S is thick in itself, and the image of a thick subset is thick, so $p_2(S)$ is thick, which means that $p_2(S)$ contains a nonemtpy open subset of $\overline{p_2(S)}$, which must then be dense. Recall that

$$p_2(S) = K = \bigcup_{g \in G} gCg^{-1}$$

So if we prove that $\overline{p_2(S)} = G$, then $p_2(S) = K$ contains a dense open subset of G and the lemma is proved. As a further reduction, if we prove that $\dim p_2(S) = \dim G$, then

$\dim \overline{p_2(S)} = \dim G$ and since $p_2(S)$ is closed and irreducible, it then follows that $\overline{p_2(S)} = G$. So it suffices to prove $\dim p_2(S) = \dim G$.

Now using Lemma 8.18, there exists $t \in C$ such that

$$\{xC \in G/C : x^{-1}tx \in C \text{ is finite}\}$$

is a finite set. That is, the fiber

$$p_2^{-1}(t) = \{(xC, t) \in G/C \times G : x^{-1}tx \in C\}$$

is finite. So the general fiber is finite, by upper semi-continuity of fiber dimension. Thus

$$\dim S = \dim p_2(S) = \dim G$$

Hence the lemma is proved. □

Remark 8.21. The proof is actually more general than the statement. Note that no properties of C were used other than it being a subgroup satisfying the conclusion of Lemma 8.18. So the proof actually shows that any closed subgroup satisfying the conclusion of Leamm 8.18 has the same property, namely that the union of all conjugates contains a dense open subset of G.

Lemma 8.22 (Closure lemma)**.** *Let G be a connected affine algebraic group acting on a variety V. Let $H \subset G$ be a parabolic subgroup, and let $W \subset V$ be a closed subset which is invariant under H. Then GW[8] is closed in V.*

Proof. Let

$$S = \{(xH, v) \in G/H \times V : x^{-1}v \in W\}$$

First, we verify that this is well defined. To do so, we need to check that the condition $x^{-1}v \in W$ does not depend on the choice of coset representative. If x, x' are both coset representatives for the same coset, then $x' = xh$ for some $h \in H$, so

$$x^{-1}v \in W \implies (x')^{-1} = (xh)^{-1} = h^{-1}x^{-1} \in hW = W$$

since W is invariant under H. So S is well defined. Now consider the morphism

$$f : G \times V \to V \qquad (x, v) \mapsto x^{-1}v$$

Since W is closed, $f^{-1}(W)$ is closed. Also consider the quotient map

$$\pi : G \times V \to G/H \times V \qquad (x, v) \mapsto (xH, v)$$

Then $S = \pi(f^{-1}(W))$. Then by Remark 8.19, since π is a surjective open map and $f^{-1}(W)$ is closed, $S = \pi(f^{-1}(W))$ is also closed. Consider the projection

$$p_2 : G/H \to V \qquad (xH, v) \mapsto v$$

Since H is parabolic, G/H is complete, so this is a closed map. So $p_2(S)$ is closed in V. Then note that $p_2(S) = GW$. □

[8]By GW, we mean any of the following equal sets.

$$GW = \bigcup_{g \in G} gW = \bigcup_{w \in W} Gw = \bigcup_{w \in W} \operatorname{orb}(w)$$

8.2 The union of Borel subgroups

Theorem 8.23. *Let G be a connected affine algebraic group.*

1. *Every element of G is contained in a Borel subgroup. That is, if B is any Borel subgroup, then*
$$G = \bigcup_{g \in G} gBg^{-1}$$

2. *Every semisimple element of G is contained in a torus.*

3. *Every unipotent element of G is contained in a connected unipotent subgroup.*

Proof. (1) Let B be a Borel subgroup. Every Cartan subgroup C is contained in some Borel subgroup, so by the Density lemma 8.20,

$$\bigcup_{g \in G} gBg^{-1}$$

contains a dense open subset of G. Note that B is a parabolic subgroup, so we may apply Lemma 8.22 with $G = V, W = H = B$, and G acting on itself by conjugation. It is clear that H is invariant under conjugation by itself.

$$G \times G \to G \qquad (x, y) \mapsto xyx^{-1}$$

So by the lemma,

$$GB = \bigcup_{g \in G} gBg^{-1}$$

is closed in G. Since we just observed that it also contains a dense open subset, it must be all of G. (2) and (3) follow immediately from (1). □

Example 8.24. In the case $G = \mathrm{GL}_n(K)$, the previous theorem just says that every invertible matrix is conjugate to an upper triangular matrix.

Remark 8.25. In part (2) of the previous theorem, the "semisimple" condition cannot be dropped. That is to say, not every element is contained in a torus. The main obstruction is the connectedness aspect. It is clear that every element is contained in some abelian subgroup, but it may not be contained in a connected abelian subgroup (which would then be contained in some torus). Concretely, then element

$$x = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(K)$$

is not contained in any connected abelian subgroup. In fact, the only connected subgroup containing x is all of $U_2(K)$, the group of upper triangular matrices, which is far from abelian. Details behind this are not especially important, so we leave them to the interested reader.

Before continuing on with corollaries of Theorem 8.23, we note a generalization, without proof.

Theorem 8.26. *Any surjective group endomorphism of a connected algebraic group keeps some Borel subgroup invariant.*

Remark 8.27. Applying this to the endomorphism of conjugation, with some work, can show that G is the union of all Borel subgroups, but we omit these details.

Now we have several major corollaries of the fact that a connected algebraic group is the union of Borel subgroups.

Corollary 8.28. *Let G be a connected algebraic group and B a Borel subgroup. Then $Z(B) = Z(G)$.*

Proof. If $x \in Z(B)$, then $x \in Z_G(B) = Z_G(G) = Z(G)$ by Corollary 8.10. Hence $Z(B) \subset Z(G)$. Conversely, suppose $x \in Z(G)$. Then $x \in B'$ for some Borel subgroup B' (by Theorem 8.23). Then B, B' are conjugate, so $B' = yBy^{-1}$ for some $y \in G$; in particular, $x = yby^{-1}$ for some $b \in B$. Then since x is central, $b = y^{-1}xy = xy^{-1}y = x$, hence $x \in B$. □

Corollary 8.29. *Let G be an algebraic group acting on an affine variety V. Let $v \in V$ such that $\operatorname{stab}(v)$ contains a maximal torus T. Then $\operatorname{orb}(v)$ is closed (in V).*

Proof. First, we deduce the general case from the connected case. Suppose the statement holds for connected groups. Then observe that $Gv = \operatorname{orb}(v)$ is the finite union of copies of G_0v [9], so since G_0v are all closed, Gv is also closed.

Now we may assume without loss of generality that G is connected. Let T be the maximal torus contained in $\operatorname{stab}(v)$, and let B be a Borel subgroup containing T. Then B is the semidirect product of T and B_U (unipotent subgroup). Let $S = Bv$ be the orbit of v with the restricted action of B on V. Then

$$S = Bv = (B_UT)v = B_Uv$$

since $T \subset \operatorname{stab}(v)$. Since S is the orbit of a unipotent group, it is closed by Proposition 6.6. Then by Lemma 8.22, since G/B is complete, Gv is closed as $Gv = GS$. □

Corollary 8.30. *Let G be an affine algebraic group.*

1. *Any conjugacy class of semsimple elements is closed.* [10]

2. *Any conjugacy class with nontrivial intersection with a Cartan subgroup is closed.*

Proof. (2) We just need to show that the conjugacy class of any element of a Cartan subgroup is closed. Let $C = Z_G(T)^0$ be a Cartan subgroup with T a maximal torus, and let $x \in C$. G acts on itself by conjugation, and under this action $T \subset \operatorname{stab}(x)$, since $x \in Z_G(T)$. Thus by Corollary 8.29, the orbit of x is closed. But the orbit of x is precisely the conjugacy class of x.

(1) Any semisimple element is contained in a torus (Theorem 8.23), so a semisimple conjugacy class meets a Cartan subgroup, so by (2) the conjugacy class is closed. □

[9] G_0 is the identity component of our not-necessarily-connected algebraic group G.

[10] Since being semisimple is preserved by conjugating, one element in a conjugacy class being semisimple is equivalent to all elements in the conjugacy class being semisimple.

Corollary 8.31. *If T is a torus in an algebraic group G, then $Z_G(T)$ is connected.* [11]

Proof. Let $x \in Z_G(T)$. Fix a Borel subgroup B, and let B' be a Borel subgroup containing x. B' acts on G/B by left multiplication.[12]

$$B' \times G/B \to G/B \qquad (b', gB) \mapsto (b'g)B$$

Let $W = \text{fix}(x) \subset G/B$ be the fixed points of x under this action. We claim W is closed. Consider

$$x : G/B \to G/B \qquad gB \mapsto (xg)B$$

Now consider the graph of this, and the diagonal in $G/B \times G/B$.

$$\Delta = \{(gB, gB) \in G/B : g \in G\}$$
$$\Gamma_x = \{(gB, xgB) \in G/B \times G/B : g \in G\} \subset G/B \times G/B$$

Since the projection map $\pi_1 : G/B \times G/B \to G/B, (g_1B, g_2B) \mapsto g_1B$ is surjective and G/B is complete, π_1 is a closed map. Hence $\pi_1(\Delta \cap \Gamma_x)$ is closed. But this is exactly $\text{fix}(x) = W$, hence W is closed. All of G acts on G/B in the same way as the above, now we consider the restriction of this action to T, and we claim that T keeps W invariant.

$$T \times G/B \to G/B \qquad (t, gB) \mapsto (tg)B$$

We need to show that for $t \in T$ and $w = gB \in W$, $tw \in W = \text{fix}(x)$. Let $t \in T$ and $w \in W$. Then using the fact that $x \in Z_G(T)$,

$$x(tw) = (xt)w = t(xw) = tw$$

Thus $tw \in \text{fix}(x) = W$, so T does keep W invariant as claimed. Thus T is a connected solvable group acting on a projective variety W, so by the Borel fixed point theorem 7.31 it has a fixed point $gB \in W$. That is, $tgB = gB$ for all $t \in T$, hence $g^{-1}Tg \subset B, T \subset gBg^{-1}$. Since $gB \in \text{fix}(x)$, we also get $g^{-1}xg \in B, x \in gBg^{-1}$. That is, x and T are in the same Borel subgroup $B'' = gBg^{-1}$.

Since $x \in Z_G(T)$, $x \in Z_{B''}(T)$. By part (5) of Theorem 7.2, $Z_{B''}(T)$ is connected, so $Z_{B''}(T) \subset Z_G(T)^0$. Thus $x \in Z_G(T)^0$, proving $Z_G(T) \subset Z_G(T)^0$, hence $Z_G(T)$ is connected. □

Remark 8.32. The previous proof shows that if G is an algebraic group and $H \subset G$ is a closed connected solvable group and $x \in Z_G(H)$, then there is a Borel subgroup B containing both x and H.

Remark 8.33. If $C = Z_G(T)^0$ is a Cartan subgroup, then $C = Z_G(T)$ as it is already connected.

[11] As a consequence, our definition of Cartan subgroups now looks rather silly. We defined a Cartan subgroup to have the form $C = Z_G(T)^0$, but because of this corollary, $Z_G(T)^0 = Z_G(T)$ for any torus T.

[12] It is worth checking that this is well-defined.

Remark 8.34. Recall that a torus T is a connected, abelian subgroup consisting of semisimple elements. The point of this remark is that the connectedness part is very important for Corollary 8.31. That is to say, the conclusion is false if T is replaced by an abelian subgroup S of semisimple elements, since such a subgroup may fail to be a torus. As an example, consider $G = \mathrm{PSL}_2(\mathbb{C})$ and the matrix [13]

$$x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Let S be the subgroup of G generated by x, so $S \cong \mathbb{Z}/4\mathbb{Z}$. One can show that $Z_G(S)$ is not connected. In particular, it is the union of diagonal matrices along with matrices of the form

$$\begin{pmatrix} 0 & z \\ -z^{-1} & 0 \end{pmatrix}$$

for $z \in \mathbb{C}$. The diagonal matrices are isomorphic to $\mathbb{G}_m$, which is connected.

Corollary 8.35. *Let G be an algebraic group and $t \in G$ be semisimple. Then every unipotent element of $Z_G(t)$ is in $Z_G(t)^0$.*

Proof. (Incomplete proof.) Let $u \in Z_G(t)$ be unipotent, and let $x = tu$. Since t, u commute, this is the Jordan decomposition of x, by uniqueness. Let B be a Borel subgroup of G containin x. Then $x = tu$ is also the Jordan decomposition in B by uniqueness, so $t, u \in B$. Since t, u commute, $u \in Z_B(t)$.

At this point, the notes assert that $Z_G(t)$ must be connected, by I don't see why this is the case. We can take T to be a torus in B containing t, and then $Z_B(T)$ is connected by a previous corollary. But then I don't see why $u \in Z_B(T)$. I don't see how to finish the proof. Apparently, we are meant to conclude that $Z_B(t)$ is connected and $u \in Z_B(t) \subset Z_G(t)^0$, hence $Z_G(t) \subset Z_G(t)^0$. $\square$

8.3 Bruhat lemma

Definition 8.36. Let G be a group and H, K subgroups. The **double coset space** $H\backslash G/K$ is the set

$$H\backslash G/K = \{HgK : g \in G\}$$

where

$$HgK = \{hgk : h \in H, k \in K\}$$

Note that two double cosets $HgK, Hg'K$ are equal if and only if there exist $h \in H, k \in K$ such that $g' = hgk$. Another way to describe double cosets is via the action of $H \times K$ on G via

$$(H \times K) \times G \to G \qquad g \mapsto hgk$$

The double coset HgK is just the orbit of g under this action.

[13] Technically elements of $\mathrm{PSL}_2(\mathbb{C})$ are equivalence classes of matrices, after quotienting out by the scalar matrices. But we may somewhat reasonably conflate matrix representatives with their equivalence class, which we do in this discussion.

Remark 8.37. Double cosets have many of the properties of normal cosets, and many much stronger properties which we don't have space to discuss.

1. Two double cosets are either disjoint or equal.
2. G is the disjoint union of all the double cosets.
3. If H is the trivial subgroup, then double cosets $H\backslash G/K$ are in bijection with cosets G/K. An analogous statement holds if K is trivial.

Lemma 8.38 (Bruhat lemma)**.** *Let G be a connected algebraic group.*

1. *Let T be a maximal torus in G and let B be a Borel subgroup with $T \subset Z_G(T) \subset B$. Let $Z = Z_G(T)$ and $N = N_G(T)$. Then the map*

$$i : Z\backslash N/Z \to B\backslash G/B \qquad ZnZ \mapsto BnB$$

   is a bijection.

2. *Any two Borel subgroups of G contain a common maximal torus.*

Remark 8.39. Before proving anything, we should verify that the double coset map is well-defined. Suppose $ZnZ = Zn'Z$ with $n, n' \in N$. Then $n' = z_1 n z_2$ for some $z_1, z_2 \in Z$. Then since $Z \subset B$, the equality $n' = z_2 n z_2$ says that $BnB = Bn'B$. Thus i is well defined.

Proof. We do not give a full proof, we just prove (1) $\implies$ (2). Let B_1, B_2 be Borel subgroups, so they are conjugate, so there is $g \in G$ so that $B_2 = gB_1g^{-1}$. Choose a maximal torus T so that $T \subset Z_G(T) \subset B_1$.

Consider the double coset BgB. By (1), this corresponds to a unique coset ZnZ under the map i, meaning $i(ZnZ) = BnB = BgB$. That is to say, there exists $n \in N_G(T)$ and $b_1, b_1' \in Z_G(T) \subset B_1$ such that $g = b_1 n b_1'$. Rearraning, we get $b_1 = g(b_1')^{-1}n^{-1}$. Since $b_1 \in Z_G(T)$,

$$b_1 T b_1^{-1} \subset B_1$$

Substituting we get

$$b_1 T b_1^{-1} = g(b_1')^{-1} n^{-1} T n b_1' g^{-1}$$

Since $n \in N_G(T)$, $n^{-1}Tn = T$, so this simplifies to

$$b_1 T b_1^{-1} = g(b_1')^{-1} T b_1' g^{-1}$$

Now because b_1' and T lie in B, $(b_1')^{-1}Tb_1' \subset B$. Hence

$$b_1 T b_1^{-1} = g(b_1')^{-1} T b_1' g^{-1} \subset gBg^{-1} = B_2$$

Hence $b_1 T b_1^{-1}$ is a maximal torus which is contained in $B_1 \cap B_2$.

In class we also gave a proof of (2) $\implies$ (1), but it involved even more technical rearrangements of group equations, so I have omitted it. □

9 Reductive and semisimple groups

Definition 9.1. Let G be an algebraic group. The **radical** $R = R(G)$ is the maximal connected solvable normal subgroup of G. The **unipotent radical** of G is the unipotent subgroup of R, denoted R_U.

Remark 9.2. The radical R is a semidirect product of a maximal torus in R with R_U, since R_U is a connected, normal subgroup of R.

Definition 9.3. An algebraic group G is **reductive** if the unipotent radical R_U is trivial. That is, R is a torus.

Definition 9.4. An algebraic group G is **semisimple** if the radical R is trivial. That is, G does not contain a normal abelian closed subgroup of positive dimension. (Obviously, semisimple $\implies$ reductive.)

Remark 9.5. If G is connected and reductive, then the radical R is contained in the center $Z(G)$. Why? Since R is normal, $N_G(R)$ is normal. Since R is connected, $N_G(R) = N_G(R)^0 = Z_G(R)^0$, hence $G = Z_G(R)^0$ so $R \subset Z(G)$.

Remark 9.6. If G is a connected algebraic group, then G/R is semisimple. In fact, G has a decomposition $G = RG_1$ where $G_1 \cong G/R$ is semisimple. This is called the **Levi decomposition**.

Example 9.7. In this example we show that $\mathrm{GL}_n(K)$ is reductive. Note that it is not semisimple, because the center is scalar matrices, which is isomorphic to $\mathbb{G}_m$ which has dimension one.

We identify $\mathrm{GL}_n(K)$ as $\mathrm{GL}(V)$ where V is an n-dimensional K-vector space. Let R be the radical. We want to show that R is a torus. Let $\{V_i\}$ be the common eigenspaces of R. This is nonempty because R is a solvable connected group (using the Borel fixed point theorem 7.31).

We claim that the subspace $\sum V_i$ is left invariant by $\mathrm{GL}(V)$. Let $g \in \mathrm{GL}(V)$ and V_i be an eigenspace for R. Let $v \in V_i$ and $r \in R$, so $rv = \alpha v$ for some $\alpha \in K$. Then

$$rgv = gg^{-1}rgv = g(g^{-1}rg)v$$

Since R is a normal subgroup, $g^{-1}rg \in R$, so $(g^{-1}rg)v = \lambda v$ for some $\lambda \in K$. Thus

$$rgv = g\lambda v = \lambda gv$$

so gv is also an eigenvector for r. That is, $\mathrm{GL}(V)$ takes $\sum V_i$ to itself. Now, because $\mathrm{GL}(V)$ acts transitively on V, the only invariant subspace is V, hence $\sum V_i = V$. Hence V is decomposable into eigenspaces for R, hence R is diagonalizable, hence R is a torus. Thus $\mathrm{GL}_n(K) \cong GL(V)$ is reductive.

Remark 9.8. The work of the previous example shows that the action of $\mathrm{GL}(V)$ on V induces an action of $\mathrm{GL}(V)$ permuting the eigenspaces V_i, thus giving a morphism from $\mathrm{GL}(V)$ to the (finite) symmetric group of permutation of the V_i. Since $\mathrm{GL}(V)$ is connected, the image of this morphism must be trivial, hence there is only one eigenspace V_i, namely all of V. The only subgroup which has all of V as an eigenspace is the subgroup $\mathbb{G}_m$ of scalar matrices, so the radical of $\mathrm{GL}(V)$ is the subgroup of scalar matrices. The Levi decomposition is $\mathrm{GL}_n(K)$ is $\mathrm{GL}(V) = R\,\mathrm{SL}_n(K)$.

Example 9.9. The groups $\mathrm{SL}_n, \mathrm{SP}_{2n}$, and SO_n are semisimple.

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